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Abstract

If $X \subset \mathbb{P}^n_{\mathbb{C}}$ is an algebraic complex projective variety, one defines the dual variety $X^* \subset (\mathbb{P}^n)^*$ as the set of tangent hyperplanes to X. The purpose of this paper is to generalise this notion when $\mathbb{P}^n$ is replaced by a quite general partial flag variety. A similar biduality theorem is proved, and the dual varieties of Schubert varieties are described.

Introduction

Let $X \subset \mathbb{P}V$ be a complex projective algebraic variety, with V a $\mathbb{C}$-vector space. If $h \in \mathbb{P}V^*$ is a hyperplane and $x \in X$ is a smooth point, we say that h is tangent to X at x if h contains the embedded tangent space of X at x. Equivalently, the intersection $X \cap h$ is singular at x. The closure of the set of all $h \in \mathbb{P}V^*$ which are tangent at some smooth point of X is denoted X^* and called the dual variety of X; for given $h \in X^*$, the closure of the set of smooth points $x \in X$ such that h is tangent at x is called the tangency locus of h.

This notion of dual varieties is a very classical one, and it is used plentifully in both classical and modern articles. The very powerfull biduality theorem, to the effect that $(X^*)^* = X$, and its corollary, which states that the tangency locus at a smooth point $h \in X^*$ is a linear space, are ubiquitous. To state only one example, this result is crucial in Zak's classification of Severi varieties, since it allows proving that the entry locus of a Severi variety is a smooth quadric [Za 93, proposition IV.2.1].

This biduality theorem deals with subvarieties of projective space, which have been studied by so many classical algebraic geometers. More recently, work has been done in a new direction which consists in considering subvarieties of other homogeneous spaces. For example, G. Faltings [Fa 81] and O. Debarre [De 96a, De 96b] have shown some connectivity theorems that hold in an arbitrary homogeneous space, E. Arrondo has proved a classification of some subvarieties of Grassmannians similar to Zak's result [Ar 99], and some topological results on low-codimensional subvarieties of some homogeneous spaces emerge in works of E. Arrondo - J. Caravantes [AC 05] and N. Perrin [Pe 07].

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Obviously, to study subvarieties of homogeneous spaces, a similar notion of dual variety and a biduality theorem are lacking. The aim of this article is to fill this gap as much as it is possible.

Since homogeneous spaces G/P are by definition projective algebraic varieties, it is certainly possible to embed them in a projective space, and therefore a subvariety $X \subset G/P$ is a fortiori a subvariety of a projective space, so that one can consider the usual dual variety of X.

However I claim that in many cases this is not the best thing to do. Let us consider an example. Let V be a $\mathbb{C}$-vector space equipped with a non-degenerate quadratic form. If $Q \subset \mathbb{P}V$ is the smooth quadric it defines, then it is well-known that $Q^* \subset \mathbb{P}V^*$ is also a smooth quadric, canonically isomorphic with Q. Now, let r be an integer and let us consider the variety $\mathbb{G}_Q(r, V)$ parametrising r-dimensional isotropic subspaces as a subvariety of a suitable projective space using Plücker embedding. Then clearly we no longer have $\mathbb{G}_Q(r, V)^* \simeq \mathbb{G}_Q(r, V)$. On the contrary, let us consider $\mathbb{G}_Q(r, V)$ as a subvariety of the Grassmannian $\mathbb{G}(r, V)$; proposition 4.1 shows that for my definition of dual varieties, $\mathbb{G}_Q(r, V) \subset \mathbb{G}(r, V)$ has a well-defined dual variety in $\mathbb{G}(r, V^*)$ which is canonically isomorphic with $\mathbb{G}_Q(r, V)$.

In fact, homogeneous spaces are often minimally embedded in projective spaces of very big dimension, so that the usual dual variety of a subvariety of a homogeneous space will happen to be very large and often untractable. A notion of dual varieties within homogeneous spaces is probably more suitable if one wants to deal with low-dimensional or low-codimension subvarieties (of course, the price to pay is that the ambient space is a bit more complicated than a projective space).

My definition of dual varieties uses a class of birational transformations called stratified Mukai flops by Namikawa [Na 06]. These are birational maps $\mu : T^*G/P \dashrightarrow T^*G/Q$ defined in terms of nilpotent orbits for some semi-simple group G and some parabolic subgroups P, Q. For given G, P, Q, if there exists such a map, then we say that G/P and G/Q allow duality. For $X \subset G/P$, we consider its conormal bundle $N^*X \subset T^*G/P$ and define the dual variety $X^Q = \pi_Q \circ \mu(N^*X) \subset G/Q$ ($\pi_Q : T^*G/Q \to G/Q$ denotes the projection) if N^*X meets the locus where μ is defined (in which case we say that X is suitable). For example, if $G = SL(V)$ and $G/P = \mathbb{P}V$, the only possibility for Q leads to $G/Q = \mathbb{P}V^*$; any proper subvariety $X \subset \mathbb{P}V$ will be suitable and $X^Q = X^*$. Another example is the fact that a Grassmannian $\mathbb{G}(r, V)$ and its dual Grassmannian $\mathbb{G}(r, V^*)$ allow duality, as one could naturally expect.

One advantage of this definition is that it uses the so-called Springer resolutions of the corresponding nilpotent orbit, which are symplectic resolutions, and this article uses heavily informations which come from the study of such resolutions [Na 06, Ch 06]. Another advantage is that it exhibits the symplectic nature of dual varieties. In fact, T^*G/P and T^*G/Q are symplectic varieties and N^*X, as a subvariety of T^*G/P, is a Lagrangian subvariety. These properties suffice to show very easily the biduality theorem 2.1 in our setting.

However, this definition also has its drawbacks. The most important is probably that it is not so much intuitive, so that given $x \in X$ and $h \in X^Q$, it is not obvious at all what the sentence "h is tangent to X at x" should mean. However, in the case of a Grassmannian, using the natural rational map

$Hom(\mathbb{C}^r, V) \dashrightarrow \mathbb{G}(r, V)$, I show that the dual variety of $X \subset \mathbb{G}(r, V)$ can be computed in terms of the usual dual variety of an adequate subvariety of $\mathbb{P}Hom(\mathbb{C}^r, V)$ (see subsection 1.6). Therefore, this is a way of understanding more easily dual varieties in the case of Grassmannians. In general however, there are two fundamental differences between our setting and usual duality.

First of all, given G, P, there may be many different Q's, or none, such that G/P and G/Q allow duality. Therefore, given suitable $X \subset G/P$, we will get a dual variety X^Q for each such Q. If one restricts to maximal parabolic subgroups, thanks to [Na 06], this difficulty disappears because for given G/P there will be at most one parabolic subgroup Q such that G/P and G/Q allow duality. Moreover, section 2 shows that one can understand all dual varieties if they are understood when P and Q are maximal parabolic subgroups. These cases are therefore called fundamental cases. They include the duality between the Grassmannian $\mathbb{G}(r, V)$ and its dual Grassmannian $\mathbb{G}(r, V^*)$, but also a duality between the two spinor varieties of a quadratic space of dimension $4p + 2$, and between the exceptional homogeneous spaces $E_6/P_1 \leftrightarrow E_6/P_6$ and $E_6/P_3 \leftrightarrow E_6/P_5$.

The second difference is that not all proper subvarieties $X \subset G/P$ will have a dual variety. Note that $X = \mathbb{P}V$ has no dual variety in $\mathbb{P}V^*$, because for any $x \in X$, no non-zero cotangent form can vanish on T_xX. From this point of view, the situation is quite similar in our setting : too big subvarieties X of G/P don't have dual varieties because for any $x \in X$ there is no generic cotangent form in T^*_xG/P which vanishes on T_xX.

In the classical setting, a hyperplane h is tangent to X at x if and only if the intersection $h \cap X$ is singular. As I already alluded to, I have not been able to give a similar geometric notion of "tangent element". The only sensible definition seemed to state that $h \in X^Q$ is tangent to X at $x \in X$ if h belongs to the image of N^*_xX under $\pi_Q \circ \mu$. Since there is an incidence variety in $G/P \times G/Q$ (the closed G-orbit), any $h \in G/Q$ still defines, exactly as in the classical situation, a subvariety $I_h \subset G/P$. Lemma 3.3 implies that if h is tangent to x at X, then the intersection $I_h \cap X$ is not transverse, but the reciprocal of this fact is false.

Section 3 deals with this matter. Corollary 3.1 states that if h is tangent to X at x, then $x \in I_h$. For $x \in X$ with X suitable, the tangent cone $\overline{T_xX} \subset G/P$ of X at x is defined in a roundabout manner as the dual variety of the variety of h's in X^Q which are tangent to X at x. Theorem 3.1 implies that $\overline{T_xX}$ is a "cone" with vertex x, where definition 3.5 generalises the classical notion of cones from subvarieties of projective space to subvarieties of fundamental homogeneous spaces.

Finally, section 4 studies dual varieties of Schubert varieties. In the classical setting, the dual variety of a linear subspace is again a linear subspace. In our setting, it is a formal consequence of the definitions that the dual variety of a Schubert variety is again a Schubert variety (see proposition 4.6 which relies on the functorial property of dual varieties given in proposition 1.3).

Let $B \subset G$ be a Borel subgroup. It turns out that the combinatorial involution $X \mapsto X^Q$ between B-stable suitable Schubert subvarieties of G/P and G/Q is no longer decreasing, as it was the case for $G/P = \mathbb{P}V$. For this reason, the description of this map is quite intricate. In the case of Grassmannians and spinor varieties, we give explicitly in terms of partitions the map $X \mapsto X^Q$, see propositions 4.8 and 4.9. This relies on a general recipy for finding X^Q when

X is a Schubert variety which is given in subsection 4.3. For the exceptional cases, this recipy theoretically defines the involution (there is only a finite number of calculations to do to compute the dual variety of a Schubert subvariety), but I will not give a more explicit description of it. As a first step, I describe a criterion for a Schubert subvariety to be suitable. Remarkably enough, this criterion can be stated in a uniform way for all the fundamental cases, using the combinatorics of some quivers studied in [Pe 06] : see theorem 4.1.

Further questions : Of course this study only gives basic properties of our generalised dual varieties : if one compares with usual dual varieties, what essentially has been proved is the biduality theorem and the computation of the dual variety of a quadric and a linear subspace. The power of the classical notion of dual varieties gives hope to me that much more can be said on this topic, including :

- Is it true that for a smooth subvariety $X \subset G/P$ one has $\dim\ X^Q \geq \dim\ X$? This question has been raised by Laurent Manivel.
- Many Fano 3-folds are defined as subvarieties of some homogeneous spaces. What are the dual varieties of these Fano 3-folds ?
- What is the dual variety of a divisor in G/P ? If this is a divisor, what is the degree of this divisor ? The answer to this question for $G/P = \mathbb{G}(2,V)$ or $G/P = E_6/P_1$ and a divisor of degree 1 has been given in [Ch 07] (the dual variety is again a divisor of degree 1).
- Classification problems : for example find all smooth varieties with dual variety a divisor of low degree.

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Contents

1 Definition of the dual variety

1.1 Notations and definition

In this subsection, I introduce the (abstract) definition of dual varieties, which allows easy proofs of general results; in subsection 1.6, an equivalent but more "down-to-earth" definition will be given in the case of Grassmannians.

Before giving this definition, which is not so intuitive, I give some "naive guesses" and explain why the corresponding notion of dual varieties would not be interesting. In this way, I hope to convince the reader that it is not possible to avoid some technicalities. Let us try our unsuccessful experiments in the case of Grassmannians.

So assume $G/P = \mathbb{G}(r, V)$ and $G/Q = \mathbb{G}(r, V^*)$ and assume $2r < \dim V$. Any element $h \in \mathbb{G}(r, V^*)$, representing a codimension r subspace of V denoted L_h, defines (at least) two subvarieties in $\mathbb{G}(r, V)$. The first (resp. the second) is the subvariety of $x \in \mathbb{G}(r, V)$ such that $L_x \subset L_h$ (resp. $\dim(L_x \cap L_h) > 0$). It will be denoted I_h (resp. $h^\perp$). Assume $X \subset \mathbb{G}(r, V)$ is a subvariety and $x \in X$. In the following, we give some naive definitions of the fact that h is tangent to X at x in terms of the intersection of X, $h^\perp$ and I_h.

Naive guess 1.1. *"h is tangent to X at x if $x \in I_h$ and the intersection $h^\perp \cap X$ is singular at x."*

This is really stupid, because if $x \in I_h$, then $h^\perp$ is singular at x, and so is the intersection $h^\perp \cap X$. So any h such that $L_x \subset L_h$ will satisfy this condition, regardless to the tangent space $T_x X$.

Naive guess 1.2. *"h is tangent to X at x if $x \in h^\perp$ and the intersection $h^\perp \cap X$ is singular at x."*

For the same reason as above, it suffices that L_h contains L_x in order that this condition holds. So if we define X^* as the set of h's satisfying the above condition, we will not have a biduality theorem. In fact, if for example $X = \{x\}$ is a point, then X^* will contain $\{h : L_h \supset L_x\}$ and $(X^*)^*$ will certainly not be reduced to $\{x\}$.

Therefore, it seems necessary to use the smooth subvariety I_h. In this case, assuming that $I_h \cap X$ is singular is not quite acurate, because I_h has codimension larger than 1, so this condition should be replaced by the fact that the intersection is not transverse :

Naive guess 1.3. *"h is tangent to X at x if $x \in I_h$ and the intersection $I_h \cap X$ is not transverse at x."*

Again, if we take $X = \{x\}$, then $X^* = \{h : L_h \supset L_x\}$, and $(X^*)^* = \{y : \dim(L_x \cap L_y) > 0\}$. So we don't have a biduality theorem.

Of course we could multiply such definitions; let us just consider one more possibility :

Naive guess 1.4. *"X^* is the intersection of the usual dual variety of X (in the Plücker embedding) with $\mathbb{G}(r, V^*)$."*

Already in case $r = 2$ and $\dim V$ even, it is easy to see that biduality will not hold. Let again $X = \{x\}$. The usual dual variety of X in the Plücker embedding is a hyperplane; therefore X^* will be a hyperplane section of $\mathbb{G}(2, V^*)$. As it is well-known, the dual variety of $\mathbb{G}(2, V^*) \subset \mathbb{P} \wedge^2 V^*$ is a hypersurface in $\mathbb{P} \wedge^2 V$. Therefore it follows that the usual dual variety of $X^* \subset \mathbb{P} \wedge^2 V^*$ will have codimension at most 2 in $\mathbb{P} \wedge^2 V$. Thus its intersection $(X^*)^*$ with $\mathbb{G}(2, V)$ cannot be a point.

I hope that the previous unsuccessfull experiments will convince the reader to accept a more conceptual definition of generalised dual varieties. Let G be a semi-simple simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. We fix $T \subset B \subset G$ a maximal torus and a Borel subgroup of G. If $P \subset G$ is a parabolic subgroup, let G/P denote the corresponding flag variety. If X is a variety, let T^*X denote its cotangent bundle; we denote $t_P : T^*G/P \to \mathfrak{g}^*$ the natual map.

Definition 1.5. *Let P, Q be parabolic subgroups of G. We say that G/P and G/Q allow duality if there is a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$ such that $t_P : T^*G/P \to \mathfrak{g}^*$ and $t_Q : T^*G/Q \to \mathfrak{g}^*$ are birational isomorphisms between the cotangent bundles and $\mathcal{O}$.*

Assume that G/P and G/Q allow duality. The birational map $t_Q^{-1} \circ t_P : T^*G/P \dashrightarrow T^*G/Q$ will be denoted μ. Let $\mathcal{O}$ be such that $t_P(T^*G/P) = t_Q(T^*G/Q) = \overline{\mathcal{O}}$. Let $X \subset G/P$ be any subvariety. Let X^{sm} denote its smooth locus and let $N^*X \subset T^*G/P$ denote the conormal bundle to X^{sm} : we have $(x, f) \in N^*X$ if and only if $x \in X^{sm}$, $f \in T_x^*G/P$, and $f_{|T_xX} = 0$.

Definition 1.6.

- *A form $f \in T^*G/P$ (resp. $f \in T^*G/Q$) is called generic if it belongs to $t_P^{-1}(\mathcal{O})$ (resp. $t_Q^{-1}(\mathcal{O})$).*
- *A subvariety $X \subset G/P$ is suitable if it is irreducible and there are generic forms in N^*X.*

- *A point x of a suitable variety X is itself suitable if there are generic forms in N^*_xX. Let X^s denotable the locus of suitable points of X.*

Remark : One could also consider reducible suitable subvarieties : they would be subvarieties such that every irreducible component is suitable; we could then define the dual variety of a reducible suitable variety as the union of dual varieties of its irreducible components.

Notation 1.7. *Let π_P denote the projection $T^*G/P \to G/P$.*

Definition 1.8. *If $X \subset G/P$ is suitable then we define $X^Q \subset G/Q$ as the image of N^*X by the rational map $\pi_Q \circ \mu$.*

In the rest of the article, P and Q will denote parabolic subgroups of a reductive simply-connected group G allowing duality. Moreover, we denote $p := \pi_P \circ \mu \ : T^*G/Q \dashrightarrow G/P$, $q := \pi_Q \circ \mu^{-1} \ : T^*G/P \dashrightarrow G/Q$ the relevant rational maps. Finally, let $\mathcal{O} \subset \mathfrak{g}^*$ be the G-orbit which is dense in $t_P(T^*G/P) = t_Q(T^*G/Q)$.

Definition 1.9.

- *Let $x \in X \subset G/P$. We say that $h \in G/Q$ is tangent to X at x if $h \in q(N^*_xX)$.*
- *If $h \in G/Q$, let I_h denote the Schubert variety of elements in G/P which are incident to h, in the sense that x is incident to h if the intersection of the stabilisors of x and h (in G) contain a Borel subgroup.*
- *As a corollary of Borel's conjugacy theorem, I_h is homogeneous under the stabilisor of h.*

The notion of tangency will be studied in more details in subsection 3. Here we only remark the following :

Fact 1.1. *If h is tangent to X at x, then the intersection $I_h \cap X$ is not transverse at x.*

The proof of this fact is postponed to section 3 : see lemma 3.3. Note that the converse does not hold in general, contrary to the case when $G/P = \mathbb{P}V$.

1.2 Fundamental cases

Definition 1.10. *Let $P, Q \subset G$ allow duality. We say that P, Q, G is a fundamental case if one of the following hold :*

- *$G = SL_n$, P and Q are the stabilisors of supplementary subspaces of $\mathbb{C}^n$.*
- *$G = Spin_{4p+2}$, P and Q are the stabilisors of supplementary (and so of different families) isotropic subspaces of $\mathbb{C}^{4p+2}$.*
- *G is of type E_6, and, with Bourbaki's notations [Bou 68, p.261] (P, Q) correspond either to the roots (α_1, α_6) or (α_3, α_5).*

If this holds, G/P and G/Q are called fundamental homogeneous spaces.

By [Na 06, theorem 6.1], these examples are all the examples of maximal parabolic subgroups allowing duality. Recall that the corresponding rational map $\mu : T^*G/P \dashrightarrow T^*G/Q$ is called a stratified Mukai flop.

Moreover, in all the other cases, the rational map $\mu : T^*G/P \dashrightarrow T^*G/Q$ (and, as we will see in subsection 2.2, the dual varieties) may be described using only fundamental stratified Mukai flops : let us recall this construction [Na 06, theorem 6.1]. Assume $P, Q \subset G$ are parabolic subgroups included in a common parabolic subgroup R. Then we have fibrations

$$\begin{array}{ccccc} G/P & & & & G/Q \\ & f_P \searrow & & \swarrow f_Q & \\ & & G/R & & \end{array}$$

with fibers R/P and R/Q. Let $U(R)$ denote the unipotent radical of R and $Z(R)$ its connected center; let $L = R/Z(R)U(R)$; $R/U(R)$ is isomorphic with a levi factor of R and L is semi-simple. Moreover, R/P and R/Q are L-homogeneous varieties : denote $\pi : R \to L$ the projection, and denote $P_L := \pi(P)$ (resp. $Q_L := \pi(Q)$) we have $R/P \simeq L/P_L$ and $R/Q \simeq L/Q_L$. Assume now that P_L, Q_L allow duality. Therefore there is a rational map $\mu_L : T^*L/P_L \dashrightarrow T^*L/Q_L$ which can be used to define the stratified Mukai flop.

In fact, let $z \in G/R$, and denote $\mathcal{F}_z := f_P^{-1}(z)$ (resp. $\mathcal{G}_z := f_Q^{-1}(z)$), and let $i_z : \mathcal{F}_z \to G/P$ (resp. $j_z : \mathcal{G}_z \to G/Q$) be the natural inclusions. We have $\mathcal{F}_z \simeq L/P_L$ and $\mathcal{G}_z \simeq L/Q_L$. Let $L_z = R_z/Z(R_z)U(R_z)$ denote the group isomorphic with L acting on $\mathcal{F}_z$ and $\mathcal{G}_z$. Because μ_L is canonical, it defines an algebraic family of rational maps $\mu_z : T^*\mathcal{F}_z \dashrightarrow T^*\mathcal{G}_z$ parametrised by G/R. Now, if α is an element of T^*G/P, say $\alpha \in T^*_xG/P$ with $x \in G/P$, then we can restrict this linear form to $T_x\mathcal{F}_{f_P(x)}$; this gives an element in the bundle $T^*\mathcal{F}_{f_P(x)}$ which we denote f_x. Finally, recall that $\pi_P : T^*G/P \to G/P$ and $\pi_Q : T^*G/Q \to G/Q$ denote the bundle projections. With these notations we have the following proposition :

Proposition 1.1. *If $f \in T^*_xG/P$ belongs to the open G-orbit, then $f_x \in T^*\mathcal{F}_{f_P(x)}$ belongs to the open $L_{f_P(x)}$-orbit, and $\pi_Q(\mu(f)) = j_{f_P(x)} \circ \pi_{Q_L} \circ \mu_x(f_x)$.*

Then, using [Ch 06, theorem 4.1], one can deduce a description of the flop $T^*G/P \dashrightarrow T^*G/Q$.

Proof : Since both maps of the proposition are equivariant, we can assume that x corresponds to the base point in G/P. If the restriction of f to $T_x\mathcal{F}_{f_P(x)}$ would belong to a closed L-stable strict subvariety of $T^*\mathcal{F}_{f_P(x)}$, then forms in the G-orbit of f would restrict to non generic forms; therefore this G-orbit could not be dense in T^*G/P.

Let $\mathfrak{u}(\mathfrak{r})$ and $\mathfrak{z}(\mathfrak{r})$ denote the nilpotent part and the centraliser of the Lie algebra $\mathfrak{r}$ of R. Let $\mathfrak{p}$ be the Lie algebra of P. Under t_P, f is mapped to an element in $\mathfrak{g}^*$ which is orthogonal to $\mathfrak{p}$ and therefore to $\mathfrak{u}(\mathfrak{r}) \oplus \mathfrak{z}(\mathfrak{r})$. It thus defines an element $\overline{f}$ in $\mathfrak{l}^*$, if $\mathfrak{l}$ denotes the Lie algebra of L. If $y \in L/Q_L$ denotes the element $\mu_L(\overline{f})$, then, by definition of the Mukai flop, $\overline{f}$ is orthogonal to $\overline{\mathfrak{q}}_y$ ($\overline{\mathfrak{q}}_y$ denotes the Lie algebra of the stabilisor of y in L). Thus it follows that f vanishes on $\mathfrak{q}_{j(y)}$, the Lie algebra of the stabiliser of $j(y)$ in G/Q. Therefore, y equals $\pi_Q \circ \mu(f)$. □

Now, [Na 06, theorem 6.1] states that for any pair (P, Q) of parabolic subgroups allowing duality, we can find a chain $(P_0 = P, P_1, \ldots, P_n = Q)$ of parabolic subgroups such that all the pairs (P_i, P_{i+1}) are as above and the corresponding pair $P_L, Q_L \subset L$ is a fundamental case. Therefore, the description of stratified Mukai flops in the fundamental cases is enough to understand all stratified Mukai flops. As we will see in section 2, the same is true as far as dual varieties are concerned.

1.3 Recallections about fundamental homogeneous spaces

We now introduce some notations and recall some results for fundamental homogeneous spaces which will be used throughout the article. In particular, we give in each case a simple way of understanding the rational map $q : T^*G/P \dashrightarrow G/Q$.

Let r and n be integers with $2r < n$. The Grassmannian parametrising r-linear subspace of a fixed n-dimensional vector space V will be denoted $\mathbb{G}(r, V)$. The dual Grassmannian, parametrising codimension r subspaces of V, will be denoted $\mathbb{G}(r, V^*)$. Let $x \in \mathbb{G}(r, V)$; it represents a linear subspace of V which will be denoted L_x. Moreover, we have a natural identification $T_x^*\mathbb{G}(r, V) \simeq Hom(V/L_x, L_x)$. If $\varphi \in Hom(V/L_x, L_x)$ is generic (that is, of rank r), then its kernel is a codimension r subspace of V containing L_x. In fact, we have $q(\varphi) = \ker \varphi$.

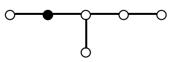
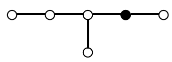
Let p be an integer. Let V be a vector space of dimension $4p + 2$, equipped with a quadratic form. In case we need a basis for V, we will take a hyperbolic one, of the form $(e_1^+, \ldots, e_{2p+1}^+, e_1^-, \ldots, e_{2p+1}^-)$, such that the quadratic form is given by $Q(\sum x_i^+ e_i^+ + \sum x_i^- e_i^-) = \sum x_i^+ x_i^-$. Recall that the variety parametrising isotropic subspaces of V of dimension $2p + 1$ has two connected components, which will be denoted $G/P = \mathbb{G}_Q^+(2p + 1, 4p + 2)$ and $G/Q = \mathbb{G}_Q^-(2p+1, 4p+2)$. As in the case of Grassmannians, for $x \in \mathbb{G}_Q^+(2p+1, 4p+2)$ and $h \in \mathbb{G}_Q^-(2p+1, 4p+2)$, we denote L_x, L_h the corresponding isotropic subspaces. The relation $x \in I_h$ amounts to $\dim(L_x \cap L_h) = 2p$. Given $x \in \mathbb{G}_Q^+(2p+1, 4p+2)$ and $L \subset L_x$ of dimension $2p$, there is exactly one $h \in \mathbb{G}_Q^-(2p + 1, 4p + 2)$ such that $L_x \cap L_h = L$: this yields a natural isomorphism between I_x and $\mathbb{P}L_x^*$.

The map q may be defined as follows. Let $x \in \mathbb{G}_Q^+(2p + 1, 4p + 2)$; the cotangent space $T_x^*\mathbb{G}_Q^+(2p + 1, 4p + 2)$ identifies with $\wedge^2 L_x$. If $\omega \in \wedge^2 L_x$ is a skew form of rank $2p$, let L_ω be its image. It is a hyperplane in L_x; therefore it defines a unique element $h \in \mathbb{G}_Q^-(2p + 1, 4p + 2)$ such that $L_x \cap L_h = L_\omega$. We have $q(\omega) = h$.

As far as the exceptional group E_6 is concerned, we denote V_i the i-th fundamental representation of E_6, so that $E_6/P_i \subset \mathbb{P}V_i$. We have $V_6 = V_1^*$ and $V_5 = V_3^*$. In terms of this embedding, an element $h \in \mathbb{P}V_1^*$ belongs to E_6/P_6 if and only if it contains the linear span of two tangent spaces $T_xE_6/P_1, T_yE_6/P_1$, for some distinct $x, y \in E_6/P_1$.

We refer to [Ch 06] for the proofs of the following results. Let $x \in E_6/P_1$. The cotangent space $T_x^*E_6/P_1$ identifies with $\mathbb{O}_\mathbb{C} \oplus \mathbb{O}_\mathbb{C}$, if $\mathbb{O}_\mathbb{C}$ denotes the algebra of complexified octonions, an 8-dimensional non-associative and non-commutative algebra over $\mathbb{C}$. This algebra is a normed algebra : there is a quadratic form $N : \mathbb{O}_\mathbb{C} \to \mathbb{C}$ such that $N(z_1 z_2) = N(z_1)N(z_2)$ for all $z_1, z_2 \in \mathbb{O}_\mathbb{C}$. The variety I_x is an 8-dimensional smooth quadric. It is convenient to denote $Z = \mathbb{C} \oplus \mathbb{O}_\mathbb{C} \oplus \mathbb{C}$ a 10-dimensional space, equipped with the quadratic form

$Q(t,z,u) = tu - N(z)$. Then I_x is the smooth quadric defined by Q and q is defined by $q((z_1,z_2)) = [N(z_1) : z_1\overline{z}_2 : N(z_2)] \in \mathbb{P}Z$ [Ch 06, theorem 3.3 and corollary 3.2].

To visualise the homogneous space E_6/P_3 (resp. E_6/P_5), we use the fact that its points parametrise projective lines included in E_6/P_1 (resp. E_6/P_6) [LM 03, theorem 4.3 p.82]. To avoid confusions between points in E_6/P_1 and E_6/P_3, we will denote the latters with greek letters. Since the marked Dynkin diagrams of E_6/P_3 and E_6/P_5 are respectively  and , we see that for $\kappa \in E_6/P_5$, $I_\kappa \simeq \mathbb{G}(2,5)$. Let us describe this isomorphism $I_\kappa \simeq \mathbb{G}(2,5)$ more explicitly, since this will be needed to describe the rational map q. If $\alpha \in E_6/P_3$, we will denote $l_\alpha \subset E_6/P_1$ the corresponding line and L_α the linear subspace it represents. By [Ch 06, proposition 3.6], the span of the affine tangent spaces $\widehat{T_x E_6/P_1}$ in V_1 for x in l_α is a 22-dimensional linear subspace in V_1 denoted S_α. Therefore, any 25-dimensional space which contains S_α defines a pencil of hyperplanes which belong to $E_6/P_6 \subset \mathbb{P}V_1^*$, that is, a point in E_6/P_5. Denoting $Q_\alpha = V_1/S_\alpha$, this shows that $I_\alpha = \mathbb{G}(3,Q_\alpha) \simeq \mathbb{G}(2,5)$. Dually, for $\beta \in E_6/P_5$, $I_\beta \simeq \mathbb{G}(2,W_\beta)$, where $W_\beta \subset V_1$ is a 5-dimensional linear subspace such that $\mathbb{P}W_\beta \subset E_6/P_1$.

A Levi factor of P contains $L' \simeq SL_2 \times SL_5$, and L_α (resp. Q_α) is the natural representation of SL_2 (resp. SL_5). These representations are usefull describing T^*E_6/P_3 : let $[e] \in E_6/P_3$ denote the base point; according to [Ch 06, propositions 3.6 and 3.7], $T^*_{[e]}E_6/P_3$ is no longer an irreducible L'-module, but there are exact sequences of L'-representations

$$0 \to L_\alpha^* \otimes \wedge^2 Q_\alpha \to T_{[e]}E_6/P_3 \to Q_\alpha^* \to 0 \tag{1}$$

$$0 \to Q_\alpha \to T^*_{[e]}E_6/P_3 \xrightarrow{\pi} L_\alpha \otimes \wedge^2 Q_\alpha^* \to 0.$$

We now describe the rational map q. Choose a base e_1^*, e_2^* (resp. $f_1, \dots, f_5$) of L_α^* (resp. Q_α). The rational map $q : T^*_{[e]}E_6/P_3 \dashrightarrow I_{[e]}$ factors through $L_\alpha \otimes \wedge^2 Q_\alpha^*$, and the induced rational map $\overline{q} : L_\alpha \otimes \wedge^2 Q_\alpha^* \dashrightarrow I_{[e]} = \mathbb{G}(2, Q_\alpha^*)$ is described as follows : let $\varphi \in L_\alpha \otimes \wedge^2 Q_\alpha^* \simeq Hom(L_\alpha \otimes \wedge^2 Q_\alpha^*)$ be generic. Its image in $\mathbb{G}(2, W_5^*)$ under $\overline{q}$ represents the linear subspace generated by

- the orthogonal for the alternate form $\varphi(e_2^*)$ of the kernel of $\varphi(e_1^*)$, and
- the orthogonal for $\varphi(e_1^*)$ of the kernel of $\varphi(e_2^*)$.

This is well-defined if and only if $\varphi(e_1^*)$ and $\varphi(e_2^*)$ have maximal rank 4 and the corresponding orthogonals are different lines in V_2^*. This is proved in [Ch 06, theorem 4.3].

1.4 Dual schemes

For some purposes (for example [Ch 07]), it may be usefull to extend the above definition of dual varieties to more general schemes. The goal of this subsection is to explain how this is possible.

Let us first define the cotangent scheme of a subscheme. So let S be an arbitrary scheme and $f : X \to Y$ a morphism above S. The cotangent scheme T^*X of X is $\mathbf{Spec}\ S^\bullet \mathcal{H}om(\Omega_{X/S}, \mathcal{O}_X)$; it is a scheme over X, equipped with a natural section, the zero section. Now f induces a natural morphism of sheaves $f^*\Omega_{Y/S} \to \Omega_{X/S}$, and so a morphism $f^*T^*Y \to T^*X$. We finally define the cotangent scheme $N^*_{X,Y}$ as the fiber above the zero section of this map.

Let G be a semi-simple Chevalley group scheme over $\mathbb{Z}$, P and Q parabolic subgroups.

Definition 1.11. *P and Q allow duality if the complex groups $P(\mathbb{C}), Q(\mathbb{C})$ do.*

If P and Q allow duality, although the moment map $T^*G/P \to \mathfrak{g}^*$ may fail to be birational in positive caracteristic, there is still a well-defined birational map $T^*G/P \dashrightarrow T^*G/Q$, defined over $\mathbb{Z}$:

Proposition 1.2. *There is a G-equivariant birational map $\mu : T^*G/P \dashrightarrow T^*G/Q$ defined over $\mathbb{Z}$.*

Proof : By [Na 06, theorem 6.1] and proposition 1.1, any pair of parabolic subgroups allowing duality is related by a chain of pairs (P, Q) of parabolic subgroups for which the birational map $T^*G/P \dashrightarrow T^*G/Q$ is locally isomorphic with a family of birational maps given by a fundamental stratified Mukai flop. It is therefore enough to check the proposition for fundamental cases. In these cases it is a consequence of the explicit description of this flop recalled in 1.3. $\square$

If S is a scheme and G, P, Q are as above, let G_S, P_S, Q_S the groups obtained by base change $S \to Spec\ \mathbb{Z}$.

Definition 1.12. *Let S be a reduced irreducible scheme, and let $f : X \to G_S/P_S$ be an irreducible closed S-subscheme. We say that X is suitable if μ is defined at the generic point of $N^*_{X,G_S/P_S}$. In this case, the dual scheme of X is the scheme-theoretic image of $N^*_{X,G_S/P_S}$ under $\pi_Q \circ \mu$.*

1.5 Functorial property of dual varieties

We come back to our setting of complex geometry. In the usual setting, if $X_1, X_2 \subset \mathbb{P}V$ are subvarieties, with $X_1 \subset X_2$, there is in general no relation of inclusion between the dual varieties of X_1 and X_2. Thus dual varieties have bad functorial properties. The only thing one can say is the following obvious result.

Proposition 1.3. *Let $P, Q \subset G$ allow duality. Let $X \subset G/P$ be suitable, $g \in G$, and $Y = g(X)$. Then Y is suitable and $g(X^Q) = Y^Q$.*

Proof : Let $x \in X^s$ and $f \in N^*_x X \subset \mathfrak{g}^*$ an element in the open G-orbit. Then ${}^t g^{-1}.f \in N^*_{g(x)}Y$ is also in the open G-orbit. Therefore, Y is suitable. Moreover, since q is equivariant, $q({}^t g^{-1}.f) = g.q(f)$. Therefore, $g(X^Q) \subset Y^Q$. By symmetry, we have also $g^{-1}(Y^Q) \subset X^Q$, so $g(X^Q) = Y^Q$. $\square$

1.6 Dual varieties in type A

In this section, I give a description of the dual variety of a subvariety $X \subset \mathbb{G}(r, V)$ using an analog of the quotient map $V \dashrightarrow \mathbb{P}V$ for Grassmannians.

If A and B are vector spaces, $Inj(A, B)$ will denote the sets of linear (resp. linear and injective) maps from A to B. Let $\varpi : Hom(\mathbb{C}^r, V) \dashrightarrow \mathbb{P}Hom(\mathbb{C}^r, V)$ denote the natural rational map, and let $\pi : \mathbb{P}Hom(\mathbb{C}^r, V) \dashrightarrow \mathbb{G}(r, V)$ map φ of rank r on its image. Dually, consider $\varpi' : Hom(V, \mathbb{C}^r) \dashrightarrow \mathbb{P}Hom(V, \mathbb{C}^r)$

and $\pi' : \mathbb{P}Hom(V, \mathbb{C}^r) \dashrightarrow \mathbb{G}(r, V^*)$ mapping φ' of rank r on its kernel. If $X \subset \mathbb{G}(r, V)$ is a subvariety, let $\overline{X}^o$ denote the set $\pi^{-1}(X)$ and $\overline{X}$ its closure in $\mathbb{P}Hom(\mathbb{C}^r, V)$.

Proposition 1.4. *Let $X \subset \mathbb{G}(r, V)$ be a suitable variety. Then $X^Q = \pi'[(\overline{X})^*]$, where $(\overline{X})^*$ is the usual dual variety of the subvariety $\overline{X} \subset \mathbb{P}Hom(\mathbb{C}^r, V)$ of a projective space.*

Proof : We fix a smooth point $x \in X$ and $\overline{f} \in \mathbb{P}Hom(\mathbb{C}^r, V)$ such that $\pi(\overline{f}) = x$, and start with two easy lemmas.

Lemma 1.2. *$\overline{X}$ is smooth at $\overline{f}$.*

Proof : In a neighbourhood of $\overline{f}$ we have $\overline{X} = \overline{X}^o$. Moreover, the map $\pi : \overline{X}^o \to X$ is locally a trivial fibration with fiber at x the smooth variety $Inj(\mathbb{C}^r, L_x)$. □

We denote $f \in Hom(\mathbb{C}^r, V)$ such that $\varpi(f) = \overline{f}$.

Lemma 1.3. *The affine tangent space $\widehat{T_{\overline{f}}\overline{X}}$ is the linear space of maps $g : \mathbb{C}^r \to V$ such that the composition $L_x \xrightarrow{f^{-1}} \mathbb{C}^r \xrightarrow{g} V \to V/L_x$ belongs to T_xX.*

Recall that for $Z \subset \mathbb{P}W$ a projective variety and $z \in Z$, the affine tangent space $\widehat{T_zZ} \subset W$ is the tangent space of the affine cone over Z at a lift of z in W.

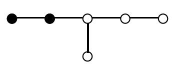
Proof : Let $\widehat{X} \subset Hom(\mathbb{C}^r, V)$ be the affine cone over $\overline{X}$. Since $\widehat{X}$ is smooth at f, any tangent vector is the direction of a curve included in $\widehat{X}$. Let $\gamma : (C, 0) \to (\widehat{X}, f)$ be a curve in $\widehat{X}$ and let $g = \gamma'(0) \in Hom(\mathbb{C}^r, V)$. Under the well-known identification of $T_x\mathbb{G}(r, V)$ with $Hom(L_x, V/L_x)$, the composition of the lemma equals $(\pi \circ \varpi \circ \gamma)'(0)$. Therefore it belongs to T_xX. By dimension count, the lemma follows. □

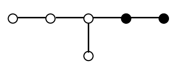
Proof of proposition 1.4 : The linear subspace $(T_f\widehat{X})^\perp \subset Hom(V, \mathbb{C}^r)$ is the set of g's such that for all $h \in T_f\widehat{X}$, the composition $\mathbb{C}^r \xrightarrow{h} V \xrightarrow{g} \mathbb{C}^r$ is traceless. Since $T_f\widehat{X}$ contains $Hom(\mathbb{C}^r, L_x)$, this means that g is induced by a morphism $\overline{g} : V/L_x \to \mathbb{C}^r$ such that $f \circ \overline{g}$ is orthogonal to $T_xX \subset Hom(L_x, V/L_x)$, by lemma 1.3. Therefore, for $h \in \mathbb{G}(r, V^*)$, we have $h \in \pi' \circ \varpi((T_f\widehat{X})^\perp)$ if and only if $h \in q(N_x^*X)$. □

2 Reduction to fundamental examples

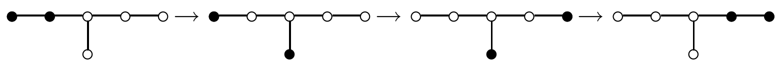
From section 1, we see that there are a lot of pairs of parabolic subgroups which allow duality. In this section, I will show that to understand all the dual varieties, it is enough to understand dual varieties for fundamental cases.

For example, the varieties corresponding to the marked diagrams





both have dimension 26. Using tables in [McG 02, p.202], we see that there is a unique nilpotent orbit of dimension 52 in $\mathfrak{e}_6$ and that the disconnected centralizer of an element of this orbit is trivial. Therefore, the two corresponding parabolic subgroups $P, Q \subset E_6$ allow duality. It may seem at first that the corresponding duality $X \subset G/P \mapsto X^Q \subset G/Q$ has to do with the exceptional geometry of E_6. However, we will see that it is not the case; indeed, X^Q can be described using dual varieties in four classical homogeneous spaces. Indeed, [Na 06, theorem 6.1] is verified in this case thanks to the sequence of parabolic subgroups



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and we will see in this section how to compute accordingly dual varieties. We will show that the computation of the dual variety X^Q for $X \subset G/P$ can be done in three steps, the first and the last in a family of spinor varieties $\mathbb{G}_Q^+(5,10)$, and the second in a family $\mathbb{P}^5$'s.

2.1 Biduality theorem

Let G be as above, $P, Q, R \subset G$ be subgroups such that P and Q allow duality, and Q and R allow duality. Then, by definition P and R also allow duality.

Theorem 2.1 (Biduality theorem). *Let $X \subset G/P$ be an suitable variety. Then X^Q is suitable and $\mu(N^*X) = N^*X^Q$. In particular, ${(X^Q)}^R = X^R$.*

If $G = SL_n$, $P = R$ is the stabilisor of a line and Q is the stabilisor of a hyperplane, we recover the usual biduality theorem.

Proof : We follow the argument of [GKZ 94, pp.27 to 30].
Let $N = \mu(N^*X) \subset T^*G/Q$. Recall that T^*G/Q is a symplectic variety. Moreover, it is proved in [GKZ 94] that $N^*X \subset T^*G/P$ is a lagrangien subvariety of T^*G/P. Let $\mathcal{O} \subset \mathfrak{g}^*$ denote the nilpotent orbit which closure is the image of T^*G/P. Since the birational morphisms $T^*G/P \overset{\sim}{\dashrightarrow} \mathcal{O} \overset{\sim}{\dashrightarrow} T^*G/Q$ are symplectic, it follows that N is also lagrangien.

Moreover, it has the property that if $(x, f) \in N$ and $\lambda \in \mathbb{C}$, then $(x, \lambda f) \in N$. This follows from the fact that the image of N^*X in $\overline{\mathcal{O}}$ is stable under multiplication by scalars. From [GKZ 94, proposition 3.1], we know that $N = N^*Z$ for $Z = \pi_Q(N) = X^Q$. Therefore, $\mu(N^*X) = N^*X^Q$ and X^Q is suitable.

Since ${(X^Q)}^R$ (resp. X^R) is the image of N^*X_Q (resp. $\mu(N^*X)$) under the rational map $T^*G/Q \dashrightarrow G/R$, these varieties are equal. $\square$

The following corollary shows that the name of biduality theorem for the above result is justified :

Corollary 2.1. *Let $P, Q \subset G$ allow duality. If $X \subset G/P$ is suitable, then X^Q is suitable and ${(X^Q)}^P = X$. Moreover, if $x \in X$ and $h \in X^Q$, then h is tangent to X at x if and only if x is tangent to X^Q at h.*

Proof : To prove that ${(X^Q)}^P = X$, it is enough to take $R = P$ in theorem 2.1, after observing that for suitable $X \subset G/P$, $X^P = X$. The second result, that h is tangent to X at x if and only if x is tangent to X^Q at h follows from the fact the first (resp. the second) affirmation means that (x, h) lies in the image by (p, π_Q) of an element in $\mu(N^*X)$ (resp. N^*X^Q). $\square$

2.2 Families of dual varieties

We consider the following situation : let $R \subset G$ be a parabolic subgroup. Let $P, Q \subset R \subset G$ be parabolic subgroups and recall notations of subsection 1.2. If $X \subset G/P$ and $z \in G/R$, denote $X_z := X \cap \mathcal{F}_z$. Assume $P_L, Q_L \subset L$ allow duality. For $z \in G/R$ and suitable $Y \subset \mathcal{F}_z \simeq L/P_L$, let $Y^{Q_L} \subset \mathcal{G}_z \simeq L/Q_L$ denote its generalised dual variety.

Theorem 2.2. *With the previous notations, assume that $P, Q \subset G$ allow duality, and also $P_L, Q_L \subset L$. If $X \subset G/P$ is suitable, then for generic $x \in X$, $X_{f_P(x)} \subset \mathcal{F}_{f_P(x)}$ is suitable. Moreover, X^Q is the closure of the union of the $X^{Q_L}_{f_P(x)}$, for such x in X.*

Proof : Let $f \in N^*_x X$ an element which G-orbit in T^*G/P is dense and set $z = f_P(x)$. We have seen in the proof of proposition 1.1 that the restriction f_x of f to $T_x\mathcal{F}_z$ is a generic element in $T^*\mathcal{F}_z \simeq T^*L/P_L$. Moreover, this restriction belongs to $N^*_x X_z$, so that X_z is suitable.

Let $q_z : T^*\mathcal{F}_z \dashrightarrow \mathcal{G}_z$ be the composition of $\mu_z : T^*\mathcal{F}_z \dashrightarrow T^*\mathcal{G}_z$ and the projection $T^*\mathcal{G}_z \to \mathcal{G}_z$. Proposition 1.1 states that $q(f) = j_z \circ q_z(f_x) \in G/Q$. Therefore it follows that $q(N^*X_{|X_z}) = j_z(X_z^{Q_L})$. The description of X^Q in the theorem follows. □

As a consequence of theorems 2.1 and 2.2, if $P = P_1 \times P_2$ and $Q = Q_1 \times Q_2$ are parabolic subgroups of $G = G_1 \times G_2$, and if $X = X_1 \times X_2$, then we have $X^Q = X_1^{Q_1} \times X_2^{Q_2}$.

3 Tangency for fundamental examples

In this section, if $x \in X \subset G/P$, I introduce a definition of the embedded tangent cone at x, $\overline{T_x X}$, which is a subvariety of G/P and a cone at x (in a suitable sense). I also introduce the cotangent variety at x, $\overline{N_x X}$, which is a subvariety of G/Q. Moreover a notion of "linear varieties" is defined and linear varieties are classified.

From now on, $P, Q \subset G$ are fundamental subgroups of G allowing duality.

3.1 A tangent element is incident

In this subsection, we prove that if $x \in X \subset G/P$ and $h \in G/Q$ is tangent to X at x (see definition 1.9), then h is incident to x (in the sense that the stabilisors of x and h contain a common Borel subgroup). This only holds in fundamental cases.

Notation 3.1. *Let $x \in \mathfrak{g}$ nilpotent. Then there exists $y, h \in \mathfrak{g}$ such that (x, y, h) is a $\mathfrak{sl}_2$-triple. For $i \in \mathbb{Z}$, let $\mathfrak{g}_i$ denote $\{X \in \mathfrak{g} : [h, X] = iX\}$. The parabolic subalgebra $\mathfrak{p}_x := \oplus_{i \geq 0}\mathfrak{g}_i$ does not depend on y and h [McG 02, theorem 3.8], and is called the canonical parabolic subalgebra of x.*

In the following lemma, I say that $\mathfrak{p} \subset \mathfrak{g}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ of fundamental type if the pair $(\mathfrak{g}, \mathfrak{p})$ is the pair of Lie algebras of groups (G, P) as in definition 1.10.

Lemma 3.1. *Let $x \in \mathfrak{g}$ and $\mathfrak{p}$ be a polarisation of x. Assume that $\mathfrak{p}$ is a maximal parabolic subalgebra of fundamental type. Then $\mathfrak{p}_x \subset \mathfrak{p}$.*

Proof : Let $\mathfrak{p}$ be a maximal parabolic subalgebra which is a polarisation of x. Let (x, y, h) be a $\mathfrak{sl}_2$-triplet, $\mathfrak{h}$ a Cartan subalgebra containing h and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ a basis of the root system such that $\forall \alpha \in \Delta, \alpha(h) \geq 0$.

We denote $\mathfrak{p}_1$ the following maximal parabolic subalgebra :

$$\mathfrak{p}_1 := \mathfrak{h} \oplus \bigoplus_{\substack{\alpha = \sum_j k_j \alpha_j \\ k_i \geq 0}} \mathfrak{g}_\alpha \quad ,$$

where i is chosen such that $\mathfrak{p}$ is conjugated to $\mathfrak{p}_1$ (such i exists because $\mathfrak{p}$ is a maximal parabolic subalgebra).

Let us prove that $x \in \mathfrak{u}(\mathfrak{p}_1)$. According to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$, we can write $x = h_x + \sum_\alpha x_\alpha$, with $h_x \in \mathfrak{h}$ and $x_\alpha \in \mathfrak{g}_\alpha$. Now, since $[h, x] = 2x$, we deduce that $h_x = 0$ and that for any root α, either $x_\alpha = 0$ or $\alpha(h) = 2$.

Claim 3.2. *If $\alpha = \sum k_j \alpha_j$ is a root, then $\alpha(h) = 2 \Longrightarrow k_i > 0$.*

Proof : This is proved by ad hoc arguments in all cases. Assume first that $\mathfrak{g} = \mathfrak{sl}_n$ and that $\mathfrak{p}$ is the stabilisor of an r-dimensional subspace. Thus $i = r$. Recall that the weighted diagram of x is by definition the list of the values $\alpha_j(h)$. The weighted diagrams of nilpotent elements in $\mathfrak{sl}_n$ are well-known; in our case, since x is a generic element of $\mathfrak{u}(\mathfrak{p})$ with $\mathfrak{p}$ conjugated to $\mathfrak{p}_1$, we have $\alpha_i(h) = \alpha_{n-i}(h) = 1$ and the other values $\alpha_j(h)$ equal 0. The equality $\alpha(h) = 2$ with $\alpha = \sum k_j \alpha_j$ amounts to $k_i + k_{n-i} = 2$, which implies $k_i = k_{n-i} = 1$.

Assume now that $\mathfrak{g} = \mathfrak{spin}_{4p+2}$. In this case, there is only one possibility for the G-orbit in $\mathfrak{spin}_{4p+2}$ of x, and $\alpha_j(h) = 1$ if and only if α_j is a spin root (ie $j \in \{2p, 2p+1\}$); otherwise $\alpha_j(h) = 0$. Therefore $\alpha(h) = 2$ implies that α is not less than the root $\alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}$, which implies the claim.

If $\mathfrak{g}$ is of type $\mathfrak{e}_6$ and $\mathfrak{p}$ corresponds to the first root, then the weighted diagram of x is $\begin{bmatrix} 1\,0\,0\,0\,1 \\ 0 \end{bmatrix}$ (see [McG 02, table p.202]). Since for all roots $\sum k_j \alpha_j$ we have $-1 \leq k_1, k_6 \leq 1$, we again have $\sum k_j \alpha_j(h) = 2 \Rightarrow k_1 = k_6 = 1$. In case $\mathfrak{p}$ corresponds to the second root, the weighted diagram is $\begin{bmatrix} 0\,1\,0\,1\,0 \\ 0 \end{bmatrix}$. The equality $\alpha(h) = 2$ for $\alpha = \sum k_j \alpha_j$ implies that $k_3 + k_5 = 2$. If $k_3 = 2$, then necessarily $k_5 \geq 1$ (see the list of roots in [Bou 68]), so we get a contradiction. Similarly $k_5 \leq 1$. So $k_3 = k_5 = 1$, and again the claim is proved. $\square$

This claim therefore proves that if $x_\alpha \neq 0$, with $\alpha = \sum k_j \alpha_j$, then $k_i > 0$. This proves that x belongs to

$$\bigoplus_{\substack{\alpha = \sum_j k_j \alpha_j \\ k_i \geq 1}} \mathfrak{g}_\alpha \ ,$$

which is readily seen to be $\mathfrak{p}_1^\perp = \mathfrak{u}(\mathfrak{p}_1)$. Thus $x \in \mathfrak{u}(\mathfrak{p}_1)$ and $\mathfrak{p}_1$ is a polarisation of x. Now, since the map $T^*G/P \to \mathfrak{g}$ is birational on its image, there is a unique polarisation of x in the conjugacy class of $\mathfrak{p}$. Therefore $\mathfrak{p} = \mathfrak{p}_1$.

Let us now show that $\mathfrak{p}_x \subset \mathfrak{p}_1$. Since $\mathfrak{p}_x \supset \mathfrak{h}$, it is the sum of $\mathfrak{h}$ and some root spaces. Now, assume $\mathfrak{g}_\alpha \subset \mathfrak{p}_x$, with $\alpha = \sum k_j \alpha_j$. This means that

$\sum k_j \alpha_j(h) \geq 0$. I claim that $k_i \geq 0$. In fact, if $k_i < 0$, then α is a negative root, so $k_j \leq 0$ for all j. We therefore have $\sum k_j \alpha_j(h) \leq k_i \alpha_i(h)$. In the proof of the above claim, we have seen that we allways have $\alpha_i(h) = 1$. So we get a contradiction.

Therefore, we have $k_i \geq 0$, and so $\mathfrak{g}_\alpha \subset \mathfrak{p}_1$. Since $\mathfrak{p}_1 = \mathfrak{p}$, we have proved that $\mathfrak{p}_x \subset \mathfrak{p}$, as claimed. □

Corollary 3.1. *If $\mathfrak{p}$ and $\mathfrak{q}$ are polarisations of the same nilpotent element x, and are maximal parabolic subalgebras of fundamental type, then they contain a common Borel subalgebra.*

Proof : They both contain the canonical parabolic subalgebra $\mathfrak{p}_x$. □

We now show, with an example, that the above corollary is wrong if one considers non maximal parabolic subalgebras.

Example 3.2. *Let $\mathfrak{g} = \mathfrak{sl}_n$. Let $x \in \mathfrak{g}$ be an element of rank 2, such that $x^3 = 0$ but $x^2 \neq 0$. Let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be the parabolic subalgebra preserving the image of x^2 and the image of x (resp. the kernel of x and the kernel of x^2). Then we have $x \in \mathfrak{u}(\mathfrak{p})$ and $x \in \mathfrak{u}(\mathfrak{q})$. However, since $\mathtt{Im}\ x \not\subset \ker x$, $\mathfrak{p}$ and $\mathfrak{q}$ are not incident.*

Proof : If $y \in \mathfrak{p}$ (resp. $y \in \mathfrak{q}$), then the commutator $[x, y]$ is strictly upper triangular for the filtration $\mathtt{Im}\ x^2 \subset \mathtt{Im}\ x \subset \mathbb{C}^n$ (resp. $\ker x \subset \ker x^2 \subset \mathbb{C}^n$). Therefore, $[x, y]$ is traceless and so $x \in \mathfrak{p}^\perp$ (resp. $x \in \mathfrak{q}^\perp$). □

The Schubert varieties I_h (recall definition 1.9) give a geometric understanding of the rational map $q : T^*G/P \dashrightarrow G/Q$:

Lemma 3.3. *Assume P and Q are maximal parabolic subgroups. Let $x \in G/P$ and $h \in G/Q$, and let f be a generic element in $T^*_x G/P$. Then $q(f) = h$ if and only if $x \in I_h$ and the cotangent form f vanishes on $T_x I_h$.*

As a consequence of the lemma, there is a unique h such that $x \in I_h$ and f vanishes on $T_x I_h$. By definition, if h is tangent to X at x, then there exists $f \in N^*_x X$ such that q is defined at f and $q(f) = h$. Thus the lemma implies that the intersection $I_h \cap X$ is not transverse at x, as was stated in fact 1.1.

Proof : Let $x \in G/P$; t_P restricts to an isomorphism between $T^*_x G/P$ and $(\mathfrak{g}/\mathfrak{p}_x)^* \subset \mathfrak{g}^*$ if $\mathfrak{p}_x$ denotes the Lie algebra of the stabilisor of x. Conversely, given $\eta \in \mathcal{O}$, $\pi_P(t_P^{-1}(\eta))$ is the unique $x \in G/P$ such that the corresponding parabolic subalgebra $\mathfrak{p}_x$ is orthogonal to η.

Let $x \in G/P$, $f \in T^*_x G/P$ generic and $\eta = t_P(f) \in \mathfrak{p}_x^\perp$, and let $h = q(f)$. The previous argument shows that h is the unique element in G/Q such that η vanishes on $\mathfrak{q}_h$. Moreover, we know by corollary 3.1 that $x \in I_h$. Note that $T_x G/P = \mathfrak{g}/\mathfrak{p}$ and $T_x I_h \simeq \mathfrak{q}_h/(\mathfrak{p}_x \cap \mathfrak{q}_h)$. Since η vanishes on $\mathfrak{p}_x$, it will vanish on $\mathfrak{q}_h$ if and only if it vanishes on $\mathfrak{q}_h/(\mathfrak{p}_x \cap \mathfrak{q}_h)$, namely, if and only if f vanishes on $T_x I_h$. □

Example 3.3. *Let $h \in G/Q$ and let $X = I_h \subset G/P$. Then X is suitable and $X^Q = \{h\}$. Moreover $p(T^*_h G/Q) = I_h$.*

Proof : First, let $x \in X$, let $f \in T_x^* X$ be generic and let $h = q(f)$. By corollary 3.1, x and h are incident, and by lemma 3.3, f vanishes on $T_x I_h$. Thus, I_h is suitable. Since G/Q is homogeneous, I_h is suitable for all $h \in G/Q$.

Let $x \in X$ and $f \in N_x^* X$ generic. Then by the above $q(f) =: h'$ is well-defined, and by lemma 3.3 again, h' is the unique element in G/Q such that $x \in I_{h'}$ and such that f vanishes on $T_x I_{h'}$. Since h satisfies these conditions, $h' = h$. Therefore, $X^Q = \{h\}$.

For the last point, we note that $p(T_h^* G/Q) = \{h\}^P = I_h$, by biduality theorem 2.1, since we have proved that $I_h^Q = \{h\}$. $\square$

3.2 Dual varieties and cones

If $X \subset \mathbb{P}V$ is included in a hyperplane represented by $h \in \mathbb{P}V^*$, then the dual variety of X, which is a subvariety of $\mathbb{P}V^*$, is a cone over h. The aim of this subsection is to prove an analogous result for our generalised dual varieties. Our first goal is to define cones.

Definition 3.4. *Let $x_1, x_2 \in G/P$*

- *x_1, x_2 are linked if there exists $h \in G/Q$ such that $x_1, x_2 \in I_h$.*
- *If $E \subset G/P$, let $I_E := \bigcap_{x \in E} I_x \subset G/Q$.*
- *If x_1, x_2 are linked, denote $L(x_1, x_2) = \bigcap_{h \in I_{\{x_1, x_2\}}} I_h$.*

In $\mathbb{P}V$, all points are linked, and $L(x_1, x_2)$ is the line through x_1 and x_2. The difference between $\mathbb{P}V$ and our general situation is that in general G does not act transitively on pairs of distinct points $x, y \in G/P$, so that $L(x, y)$ may depend, up to isomorphism, on x and y. However, cones are defined in perfect analogy :

Definition 3.5. *Let $X \subset G/P$ and $x \in X$. Then X is a cone over x if for all $y \in X$, x and y are linked and $L(x, y) \subset X$.*

An equivalent definition is that for generic $y \in X$ the same condition holds, as will be clear from the following description of $L(x, y)$:

Proposition 3.2. *Let $x \neq y \in G/P$. We have :*

- *If $G/P = \mathbb{G}(r, V)$, then (x, y) are linked if and only if $codim_V(L_x + L_y) \geq r$, in which case $L(x, y) = \mathbb{G}(r, L_x + L_y)$.*
- *If $G/P = \mathbb{G}_Q^+(2p+1, 4p+2)$, then (x, y) are linked if and only if we have $\dim(L_x \cap L_y) = 2p - 1$, in which case $L(x, y) = \{z : L_z \supset L_x \cap L_y\} \simeq \mathbb{P}^1$.*
- *If $G/P = E_6/P_1$, then (x, y) are allways linked. In case a line passes through x and y in E_6/P_1, then $L(x, y)$ is this line; otherwise, there is a unique smooth 8-dimensional quadric through x and y, and $L(x, y)$ is this quadric.*
- *If $G/P = E_6/P_3$, then (x, y) are linked if and only if there is a $\mathbb{G}(2, 5)$ through them. If $\dim(L_x \cap L_y) = 1$ then $L(x, y)$ is equal to $\mathbb{G}(2, L_x + L_y) \simeq \mathbb{P}^2$, otherwise $L(x, y) \simeq \mathbb{G}(2, 5)$.*

In this proposition, for the two exceptional cases, I use the minimal projective homogeneous embedding $E_6/P_i \subset \mathbb{P}V_i$. For example, in the case of E_6/P_3, the condition that there is a $\mathbb{G}(2,5)$ through x and y means that there is a linear 10-dimensional subspace $W \subset V_3$ containing x and y and such that $\mathbb{P}W \cap E_6/P_3$ is projectively isomorphic with a Grassmanian $\mathbb{G}(2,5)$ in its Plücker embedding. Recall also that E_6/P_3 parametrises projective lines in V_1 which are included in E_6/P_1. For $x \in E_6/P_3$, the corresponding 2-dimensional subspace of V_1 has been denoted L_x.

Proof : The first case follows directly from the definition. In the second case, one only has to note that if there exists $h \in G/Q$ such that $(x,h),(y,h)$ are incident, then $\dim(L_x \cap L_h) = \dim(L_y \cap L_h) = 2p$, so $\dim(L_x \cap L_h \cap L_y) = 2p-1$.

For the exceptional cases one oviously has to use the geometry of the involved homogeneous spaces. Let us first consider E_6/P_1. For all $x \in E_6/P_1$, I_x is a smooth 8-dimensional quadric. Moreover, for any $x \neq y \in E_6/P_1$, the intersection of the two quadrics I_x and I_y is either a point or a $\mathbb{P}^4$. In fact, this was proved in [Za 93, propositions IV.3.2 and IV.3.3] in the context of Severi varieties, but also follows easily from the fact that there are three E_6-orbits in $E_6/P_1 \times E_6/P_1$ [CMP 06, proposition 18]. Given $x, y \in E_6/P_1$, we can have $x = y$, $x \neq y$ and there is a line through x and y, or there is no line through x and y. This describes the three orbits in $E_6/P_1 \times E_6/P_1$. In the degenerate case when a line passes through x and y, $I_x \cap I_y$ is thus isomorphic with $\mathbb{P}^4$. Dually, the intersection of all the I_h for h in this $\mathbb{P}^4$ is a linear space (indeed, $x \in I_h$ if and only if $x \in E_6/P_1 \subset \mathbb{P}V_1$ is orthogonal to $\widehat{T_h E_6/P_6} \subset V_6 = V_1^*$) and contains x and y; a direct computation of dimension shows that it is exactly the line through x and y. In the generic case, $I_x \cap I_y = \{h\}$; therefore $L(x,y) = I_h$ is the unique 8-dimensional quadric through x and y.

Let $\alpha, \beta \in E_6/P_3$ be linked, and denote $\kappa \in E_6/P_5$ an element such that $\alpha, \beta \in I_\kappa$. According to subsection 1.2, α and β represent 2-dimensional subspaces of a 5-dimensional subspace of V_1 denoted W_κ; we have denoted L_α, L_β these spaces.

Assume first that $\dim(L_\alpha \cap L_\beta) = 1$. It is proved in [Ch 06, proposition 3.6] that the linear span of all the affine tangent spaces at the points of the projective plane generated by L_α and L_β is 24-dimensional and equal to the span of affine tangent spaces at points in $l_\alpha \cup l_\beta$. Thus (α, β) defines a projective plane in E_6/P_6 and also in E_6/P_5. Moreover $I_{\alpha,\beta} = I_{\mathbb{G}(2,L_\alpha+L_\beta)} \simeq \mathbb{P}^2$ and $L(\alpha,\beta) = \mathbb{G}(2, L_\alpha + L_\beta) \simeq \mathbb{P}^2$.

Assume finally that L_α and L_β don't meet. Let $L \subset L_\alpha \oplus L_\beta$ be any 3-dimensional subspace; the linear span S_L of the affine tangent spaces at points of $\mathbb{P}L$ is again 24-dimensional, and any element in $I_{\alpha,\beta}$ must contain it. Assume that $I_{\alpha,\beta}$ contains two points $\kappa, \lambda \in G/Q$. These points would correspond to codimension 2 subspaces L_κ, L_λ of V_1 containing S_L; therefore L_κ and L_λ would be contained in a common hyperplane of V_1. Since $\alpha, \beta \in I_{\kappa,\lambda}$, by the case considered above, this would in turn imply that L_α and L_β meet in dimension 1, which we have excluded. Therefore we have proved that $I_{\alpha,\beta} = \{\kappa\}$, so $L(\alpha,\beta) = I_\kappa$ is isomorphic with $\mathbb{G}(2,5)$. □

Theorem 3.1. *Let $h \in G/Q$ and let $X \subset G/P$ such that $X \subset I_h$. Then X is suitable and X^Q is a cone over h.*

Remark : In fact, as the proof will show, in all the cases but in type A, a stronger result holds : for any $k \in X^Q$, there is a certain homogeneous subvariety $q(\mathbb{C}.f + N_x^* I_h) \subset G/Q$, of type given by lemmas 3.4, 3.5, 3.6, and 3.7, containing (eventually stricly) $L(h,k)$, and included in X^Q. Although the idea of proof of this theorem is uniform, this proof unfortunately ends up with a case by case analysis.

Proof : If $x \in X$, then $N_x^* X$ contains $N_x^* I_h$ on which q is well-defined generically, so X is suitable. Assume $X \subset I_h$ and let k be a generic element in X^Q. By definition of X^Q there is an element $x \in X$ and $f \in N_x^* X$ such that $k = q(f)$. Since $x \in X \subset I_h$, we have $h \in I_x$. By corollary 3.1, $k \in I_x$; therefore h and k are linked. Moreover, we have $f \notin N_x^* I_h$ (otherwise we would have $q(f) = h$). Therefore it follows from the inclusion $q(\mathbb{C}.f + N_x^* I_h) \subset X^Q$ and the following lemmas 3.4, 3.5, 3.6 and 3.7 that $L(h,k) \subset X^Q$. □

Lemma 3.4. *Let $x \in \mathbb{G}(r,V)$, $h \neq k \in \mathbb{G}(r,V^*)$ such that $h,k \in I_x$. Let $f \in N_x^* I_h$ such that q is defined at f. Then $q(\mathbb{C}.f + N_x^* I_k) = L(h,k)$.*

Proof : Let (e_i) be a base of V and (e_i^*) the dual base. Up to the action of $SL(V)$, we may assume that L_x is the span of $e_1, \dots, e_r$, L_h is the span of $e_{n-r+1}^*, \dots, e_n^*$, L_k that of $e_{n-r+1-l}^*, \dots, e_{n-r}^*, e_{n-r+1}^*, \dots, e_{n-l}^*$, and finally that $f \in N_x^* I_h \simeq Hom(L_x^*, L_h)$ is defined by $f(e_j^*) = e_{n-r+j}^*$. Since $N_x^* I_k = Hom(L_x^*, L_k)$, a straightforward computation proves the lemma. □

Lemma 3.5. *Let $x \in \mathbb{G}_Q^+(2p+1, 4p+2)$, $h \neq k \in \mathbb{G}_Q^-(2p+1, 4p+2)$ such that $h,k \in I_x$. Let $f \in N_x^* I_h$ such that q is defined at f. Then $q(\mathbb{C}.f + N_x^* I_k) \simeq \mathbb{P}^{2p-1}$.*

This lemma implies theorem 3.1 in this case since $q(\mathbb{C}.f + N_x^* I_k)$ is a linear space containing h and k, and will therefore contain the line through h and k.

Proof : We may assume that x represents the isotropic subspace L_x generated by $e_1^+, \dots, e_{2p+1}^+$. Since L_k meets L_x along a hyperplane, we may further assume that this hyperplane is generated by $e_2^+, \dots, e_{2p+1}^+$. We therefore have $N_x^* I_k = \wedge^2 \langle e_2^+, \dots, e_{2p+1}^+ \rangle \subset \wedge^2 L_x = T_x^* \mathbb{G}_Q^+(2p+1, 4p+2)$. Let $f \in T_x^* \mathbb{G}_Q^+(2p+1, 4p+2)$; since $f \notin N_x^* I_k$ (otherwise we would have $h = k$), the class of f modulo $N_x^* I_h$ is the same as that of some form $e_1^+ \wedge e$, with $e \in \langle e_2^+, \dots, e_{2p+1}^+ \rangle$, and we may assume that $e = e_2^+$.

Recall that $I_x \simeq \mathbb{P} L_x^*$: I claim that $q(\mathbb{C}.f + N_x^* I_k)$ is the orthogonal of e_2^+ in $\mathbb{P} L_x^*$. In fact, let $\wedge^p(\mathbb{C}.f + N_x^* I_k) \subset \wedge^{2p} L_x \simeq L_x^*$ be the linear span of all the forms in $\wedge^{2p} L_x$ which can be written as a wedge product of p forms in $\mathbb{C}.f \oplus N_x^* I_k$. We have $\wedge^p(\mathbb{C}.f + N_x^* I_k) \subset (e_2^+)^\perp$; therefore $q(\mathbb{C}.f + N_x^* I_k) \subset \mathbb{P}(e_2^+)^\perp$.

On the other hand, let $f_0 = \Sigma_{i=1}^p e_{2i-1}^+ \wedge e_{2i}^+$; we have $f_0^{\wedge(p-1)} = \Sigma_{i=1}^p {e_{2i-1}^+}^* \wedge {e_{2i}^+}^* \wedge e_{2p+1}^*$, from which is follows that the rational map $\mathbb{C}.f + N_x^* I_k \dashrightarrow \mathbb{P}(e_2^+)^\perp, g \mapsto [g^{\wedge p}]$ is submersive at f_0, which implies the claim and the lemma. □

Lemma 3.6. *Let $x \in E_6/P_1$, $h \neq k \in E_6/P_6$ such that $h,k \in I_x$. Let $f \in N_x^* I_h$ such that q is defined at f. If there passes a line through h and k in E_6/P_6 then $q(\mathbb{C}.f + N_x^* I_k) \simeq \mathbb{P}^4$, otherwise $q(\mathbb{C}.f + N_x^* I_k) = I_x$*

Proof : We adopt the same strategy of proof as for lemma 3.5. Let $x \in E_6/P_1$ be fixed. In subsection 1.2, we saw that T_x^*X identifies with $\mathbb{O}_{\mathbb{C}} \oplus \mathbb{O}_{\mathbb{C}}$, I_x with the projective quadric in $\mathbb{P}(\mathbb{C} \oplus \mathbb{O}_{\mathbb{C}} \oplus \mathbb{C})$ defined by $tu - N(z) = 0$. We can assume that $k \in I_x$ is the class of $(0,0,1)$. Therefore, $N_x^* I_k = \overline{q^{-1}(k)} = \{(0,z) : z \in \mathbb{O}_{\mathbb{C}}\}$.

Write $f = (z_0, z_1)$. Since $q((z_0,z_1)) = [N(z_0) : z_0\overline{z}_1 : N(z_1)]$, there will be a line through $q(f)$ and k in the quadric I_x if and only if $N(z_0) = 0$. If this occurs, then

$$q(\mathbb{C}.f + N_x^* I_k) = \{[(0,u,t)] : t \in \mathbb{C}, u \in L(z_0)\},$$

where $L(z_0)$ denotes the set of right multiples of z_0 : $L(z_0) = \{z_0 z : z \in \mathbb{O}_{\mathbb{C}}\}$. It is a linear subspace of $\mathbb{O}_{\mathbb{C}}$ of dimension 4, so $q(\mathbb{C}.f + N_x^* I_k)$ is isomorphic with $\mathbb{P}^4$, as desired. If $N(z_0) \neq 0$, then left multiplication by z_0 is invertible, so that $q : \mathbb{C}.f + N_x^* I_k \dashrightarrow I_x$ is dominant, and the lemma again holds. □

Lemma 3.7. *Let $\alpha \in E_6/P_2$, $\kappa, \lambda \in E_6/P_5$ such that $\kappa, \lambda \in I_\alpha$. Let $f \in N_\alpha^* I_\kappa$ such that q is defined at f. Then $q(\mathbb{C}.f + N_\alpha^* I_\kappa) = I_\alpha$.*

Proof : We fix $\alpha \in E_6/P_2$. Let $f_1^*, \ldots, f_5^*$ be a base of Q_α^* and assume that κ corresponds to the linear subspace generated by f_4^*, f_5^*. Recall that there is a natural surjective map $\pi : T_\alpha^* E_6/P_2 \to Hom(L_\alpha^*, \wedge^2 Q_\alpha^*)$. Moreover, $\pi(N_\alpha^* I_\kappa) = Hom(L_\alpha^*, L) \subset Hom(L_\alpha^*, \wedge^2 Q_\alpha^*)$, where $L \subset \wedge^2 Q_\alpha^*$ is generated by $f_1^* \wedge f_4^*, f_1^* \wedge f_5^*, f_2^* \wedge f_4^*, f_2^* \wedge f_5^*, f_3^* \wedge f_4^*, f_3^* \wedge f_5^*, f_4^* \wedge f_5^*$ (for example, this follows from the fact that for any $\varphi \in Hom(L_\alpha^*, L)$, $\overline{q}(\varphi)$, if defined, equals κ).

Let $M \subset \wedge^2 Q_\alpha^*$ be generated by $f_1^* \wedge f_2^*, f_1^* \wedge f_3^*, f_2^* \wedge f_3^*$, so that $L \oplus M = \wedge^2 Q_\alpha^*$; the class of $\pi(f)$ modulo $\pi(N_\alpha^* I_\kappa)$ is the class of a unique $\overline{f} \in Hom(L_\alpha^*, M)$. Assume first that the rank of $\overline{f}$ is 1. We can therefore assume that $\overline{f}(e_1^*) = f_1^* \wedge f_2^*$ and $\overline{f}(e_2^*) = 0$, where e_1^*, e_2^* is a suitable basis of L_α^*.

In the array below we give, for $\omega \in \wedge^2 Q_\alpha^*$, the value of the derivative $d\overline{q}_{\overline{f}}(\varphi)$, for $\varphi : L_\alpha^* \to \wedge^2 Q_\alpha^*$ given by $\varphi(e_1^*) = \omega$ and $\varphi(e_2^*) = 0$:

$$\begin{array}{llll}
f_1^* \wedge f_2^* \mapsto 0 & f_1^* \wedge f_3^* \mapsto 0 & f_1^* \wedge f_4^* \mapsto 0 & f_1^* \wedge f_5^* \mapsto 0 \\
f_2^* \wedge f_3^* \mapsto 0 & f_2^* \wedge f_4^* \mapsto f_3^* \wedge f_4^* & f_2^* \wedge f_5^* \mapsto f_1^* \wedge f_4^* + f_3^* \wedge f_5^* & \\
f_3^* \wedge f_4^* \mapsto 0 & f_3^* \wedge f_5^* \mapsto 0 & f_4^* \wedge f_5^* \mapsto f_1^* \wedge f_5^*. &
\end{array}$$

The following gives similar values for φ defined by $\varphi(e_1^*) = 0$ and $\varphi(e_2^*) = \omega$:

$$\begin{array}{llll}
f_1^* \wedge f_2^* \mapsto 0 & f_1^* \wedge f_3^* \mapsto 0 & f_1^* \wedge f_4^* \mapsto 0 & f_1^* \wedge f_5^* \mapsto 0 \\
f_2^* \wedge f_3^* \mapsto 0 & f_2^* \wedge f_4^* \mapsto f_2^* \wedge f_3^* & f_2^* \wedge f_5^* \mapsto f_3^* \wedge f_4^* + f_1^* \wedge f_2^* & \\
f_3^* \wedge f_4^* \mapsto 0 & f_3^* \wedge f_5^* \mapsto 0 & f_4^* \wedge f_5^* \mapsto f_1^* \wedge f_4^*. &
\end{array}$$

It follows from these computations that $\overline{q}(\mathbb{C}.\overline{f} + \pi(N_\alpha^* I_\kappa))$ has dimension at least 6, so $\overline{q}(\mathbb{C}.\overline{f} + \pi(N_\alpha^* I_\kappa)) = I_\alpha$ in this case.

In case $\overline{f}$ has rank 2, the dimension of $\overline{q}(\mathbb{C}.\overline{f} + \pi(N_\alpha^* I_\kappa))$ will not vary if $\overline{f}$ is replaced by $g.\overline{f}$, where $g \in SL(L_\alpha) \times SL(Q_\alpha)$ preserves κ. Using a $\mathbb{C}^*$-action we can degenerate $\overline{f} \in Hom(L_\alpha^*, M)$ to some element $\overline{f}_0$ of rank one, for which we have already seen that $\dim \overline{q}(\mathbb{C}.f_0 + \pi(N_\alpha^* I_\kappa)) = 6$. Since this dimension is lower semi-continuous, we have $\dim \overline{q}(\mathbb{C}.\overline{f} + \pi(N_\alpha^* I_\kappa)) = 6$ and the lemma is proved. □

3.3 The cotangent space and the tangent cone of a variety

Definition 3.6. *Let $x \in X$ be suitable.*

- *The embedded cotangent space of X at x is $\overline{N_x X} := q(N_x^* X) \subset G/Q$.*
- *The embedded tangent space of X at x is $\overline{T_x X} = \overline{N_x X}^P$.*
- *$X \subset G/P$ is a linear subvariety if $\overline{T_x X}$ does not depend on x suitable in X.*

Remarks :

- The notion of (co)-tangent space (and therefore of linear varieties) of $X \subset G/P$ could be defined for non maximal parabolic P, but then it would depend on the choice of a parabolic subgroup Q.
- An equivalent definition of linear subvarieties is that $\overline{N_x X}$ does not depend on suitable x in X, since $\overline{N_x X} = \overline{T_x X}^Q$.
- By definition, $X^Q = \overline{\cup_{x \in X^s} \overline{N_x X}}$.
- In projective spaces, the tangent cone is the usual embedded tangent space and linear varieties are linear subspaces. Linear varieties will be classified in the next subsection.

Example 3.7. *Let $x \in G/P$ and $X = \{x\}$. Then $\overline{T_x X} = \{x\}$.*

Proof : In fact, $\overline{N_x X} = q(T_x^* G/P) = X^Q$, so this follows from theorem 2.1. $\square$

Lemma 3.8. *For $x \in X^s$, $\overline{T_x X}$ is a cone over x and therefore $x \in \overline{T_x X}$.*

Proof : In fact, for $x \in X^s$, we have $\overline{N_x X} \subset I_x$, so $\overline{T_x X} = \overline{N_x X}^P$ is a cone over x by theorem 3.1. $\square$

3.4 Linear subvarieties

In this subsection, we classify linear subvarieties.

Proposition 3.3. *The following array gives the list of all linear subvarieties :*

G/P	*Linear varieties*
$\mathbb{G}(r,n)$	$\mathbb{G}(r,p),\ r \leq p \leq n$
$\mathbb{G}_Q^+(2p+1,4p+2)$	$\{pt\}; I_h, h \in \mathbb{G}_Q^-(2p+1,4p+2)$
E_6/P_1	$\{pt\}; I_h, h \in E_6/P_6$
E_6/P_2	$\{pt\}; I_\kappa, \kappa \in E_6/P_5$

Proof : Let $X \subset G/P$ be linear. First, we prove that $\forall x \in X, \overline{T_x X} = X$, and that X^Q is linear. Let $x \in X^s$. Then $X^Q = \overline{\cup_{y \in X^s} \overline{N_y X}} = \overline{N_x X}$, since for all $y \in X^s$, $\overline{N_y X} = \overline{N_x X}$. Therefore, $X = \overline{N_x X}^P = \overline{T_x X}$ by corollary 2.1. Let $h \in X^Q$ and $x \in X$. Then, by biduality theorem again, $x \in \overline{N_h X^Q}$ if and only if $h \in \overline{N_x X} = X^Q$. Therefore, $\overline{N_h X^Q} = X$ and X^Q is linear and the claim is proved. Since $X = \overline{T_x X}$ for all $x \in X$, X is a cone over all of its points by theorem 3.1.

We finish the proof case by case. In the case of Grassmannians, if we denote $W = \sum_{x\in X} L_x$, since X is a cone over all of its points, we have $\mathbb{G}(r, W) \subset X$, and so $X = \mathbb{G}(r, W)$.

In the case of spinor varieties, any $x, y \in X$ must be linked, which implies that the line through x and y is in X, so X is a linear subspace. As a consequence of the following proposition 4.9, the only linear subspaces which dual variety is again a linear subspace are the point and maximal linear subspaces. Since we have seen that X^Q must be a linear variety, the proposition follows in this case.

Let $X \subset E_6/P_1$ be linear. Let $h \in X^Q$. If there are two points $x, y \in X$ such that there is no line through x and y, then by lemma 3.6 $L(x, y) = I_h$. Since $X \subset I_h$ and $L(x, y) \subset X$, we have $X = I_h$ (and X^Q is a point).

Otherwise, by theorem 3.1, X is a linear subspace. If X^Q is not a linear subspace, by the argument above, $X = (X^Q)^P$ is a point. Assume now that both X and X^Q are linear subspaces, not reduced to a point. By lemma 3.6, X^Q and $X = (X^Q)^P$ contain a $\mathbb{P}^4$. But this implies that $X^Q \subset I_X$ is at most 1-dimensional (see the proof of theorem 3.1), and we get a contradiction.

Let finally $X \subset E_6/P_3$ be linear. Assume X is not reduced to a point. Let $h \in X^Q$; we have $X \subset I_h$. On the other hand, since X is not a point, by lemma 3.7, it must contain I_h. Therefore, $X = I_h$. □

4 Examples of dual varieties

4.1 Dual varieties of isotropic Grassmannians

Let V be a vector space, $B : V \to V^*$ a bilinear form. If $\epsilon = \pm 1$ and ${}^tB = \epsilon B$, we say that B is ϵ-symmetric. Assume that this is the case. Let r be an integer; we consider the variety $\mathbb{G}_B(r, V)$ of isotropic subspaces of V of dimension r. The aim of this subsection is to describe the dual of $\mathbb{G}_B(r, V)$ in $G(r, V^*)$ in case $2r < \dim\ V$ (the other cases would be similar). Note that we don't assume that B is an isomorphism.

We have a rational map $\mathbb{G}_B(r, V) \dashrightarrow \mathbb{G}(r, V^*)$ which maps a linear subspace to its orthogonal, and which is well-defined at the point α if and only if L_α does not meet the kernel of B. Assuming there are such points, we call co-isotropic Grassmannian the image of this rational map.

Proposition 4.1. *Assume $\epsilon = 1$. Then $\mathbb{G}_B(r, V)$ is suitable if and only if and $r \leq \mathtt{rk}(B)$. In this case, the dual variety of the isotropic Grassmannian $G_B(r, V)$ is the co-isotropic Grassmannian.*

Proposition 4.2. *Assume $\epsilon = -1$. Then $\mathbb{G}_B(r, V)$ is suitable if and only if r is even and $r \leq \mathtt{rk}(B)$. In this case, the dual variety of the isotropic Grassmannian $G_B(r, V)$ is the co-isotropic Grassmannian.*

Proof : We prove propositions 4.1 and 4.2 simultaneously. Let $x \in \mathbb{G}_B(r, V)$ be generic. Under the natural isomorphism $T_x\mathbb{G}(r, V) \simeq Hom(L_x, V/L_x)$, we have the inclusion $T_x\mathbb{G}_B(r, V) \supset Hom(L_x, L_x^\perp/L_x)$, where $L_x^\perp$ denotes the orthogonal of L_x with respect to B. It follows that if codim $L_x^\perp < r$, then $N_x^*\mathbb{G}_B(r, V)$ does not meet the open orbit in $T_x^*\mathbb{G}_B(r, V)$. If $r > \mathtt{rk}(B)$, this occurs for all $x \in \mathbb{G}_B(r, V)$, hence $\mathbb{G}_B(r, V)$ is not suitable.

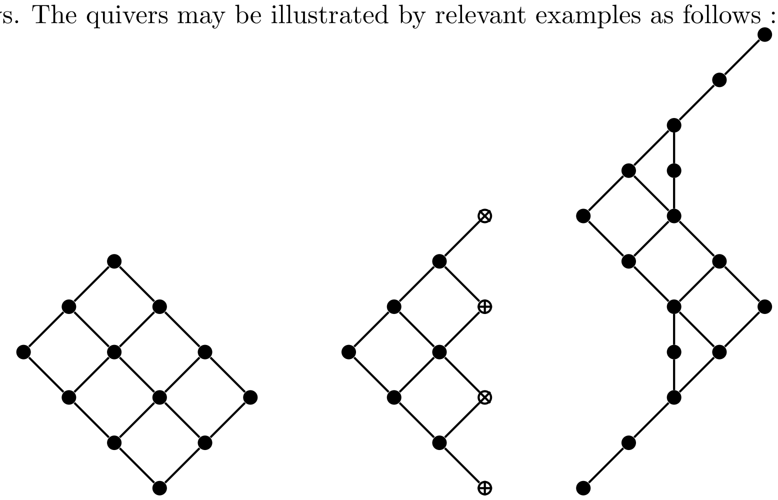
Assume $r \leq \mathtt{rk}(B)$. Now, let $x \in \mathbb{G}_B(r, V)$ such that codim $L_x^\perp = r$. Denote $Q_x = V/L_x$; we have a morphism $Q_x \to L_x^*$, induced by B. Clearly, $T_x\mathbb{G}_B(r, V) \subset Hom(T_x, Q_x)$ is the subspace of linear maps such that the composition $L_x \to Q_x \to L_x^*$ is $(-\epsilon)$-symmetric. Therefore, the normal space of $\mathbb{G}_B(r, V)$ at x identifies with ϵ-symmetric maps $L_x^* \to L_x$. Since $\mathbb{G}_B(r, V)$ will be suitable if and only if there are such maps of rank r, this occurs in all cases if $\epsilon = 1$ and exactly when r is even when $\epsilon = -1$.

Now, the computation of the dual variety is straightforward : since we have already remarked that $T_x\mathbb{G}_B(r, V) \supset Hom(L_x, L_x^\perp/L_x)$, the image of a generic conormal form at x under the rational map $q : T_x^*\mathbb{G}(r, V) \dashrightarrow G(r, V^*)$ is the element in $\mathbb{G}(r, V^*)$ corresponding to $L_x^\perp$. $\square$

4.2 Schubert varieties and quivers in the fundamental case

In this subsection, I recall that to a cominuscule homogeneous space one can naturally associate a quiver, such that Schubert cells are parametrised by some subquivers. I also recall the Hasse diagram of a representation, and show how the quiver of a cominuscule homogeneous space can be identified with the Hasse diagram of a tangent space. This identification is due to Nicolas Perrin and Laurent Manivel. Then, I show that this identification behaves well as far as Schubert subvarieties are concerned. Finally, I extend these results to E_6/P_3, which is not a cominuscule homogeneous space.

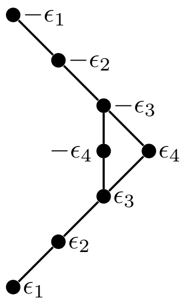
The quiver of a cominuscule homogeneous space has been first introduced by N. Perrin [Pe 06, definition 3.2]; here we use the slightly different definition [CMP 06, definition 2.1]. Recall that $\mathbb{G}(r, V), \mathbb{G}_Q^+(2p+1, 4p+2)$ and E_6/P_1 are cominuscule spaces (in fact even minuscule). The quiver is defined using a reduced expression of $w_{G/P}$, the shortest element in the class of w_0 in W/W_P (w_0 is the longest element in W). Choose a reduced expression $w_{G/P} = s_{\beta_1} \cdots s_{\beta_N}$, with $N = \dim G/P$; the vertices of the quiver $Q_{G/P}$ are in bijection with $[1, \ldots, N]$, and we refer to [CMP 06, definition 2.1] for the definition of the arrows. The quivers may be illustrated by relevant examples as follows :



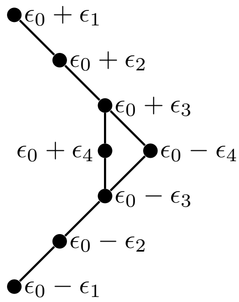
$\mathbb{G}(3, 7)$ $\qquad$ $\mathbb{G}_Q^+(5, 10)$ $\qquad$ E_6/P_1

In these pictures, all arrows are going down. Moreover, we will use the definition of height of a vertex of such a quiver. More or less by definition (see [Pe 06, definition 4.7]), it is the height of the vertex in the above drawing, where by convention the lowest vertex has height 1 (so the highest vertex has height respectively 6,7,11 for $\mathbb{G}(3,7), \mathbb{G}_Q^+(5,10), E_6/P_1$).

Later we will have to identify this quiver with a Hasse diagram. Let V be a representation of a semi-simple group Λ. Let us recall that the Hasse diagram of V is a quiver defined as follows. The vertices of this quiver are the weights of V, and there is an arrow from λ_1 to λ_2 if and only if $\lambda_2 - \lambda_1$ is a simple root. For example, the Hasse diagram of the 8-dimensional representation of $Spin_8$ is given on the left :



$Spin_8$



Roots for $\mathbb{Q}^8$

Proposition 4.3. *Let G/P be cominuscule and let $x \in G/P$ be the base point. Let Λ be a Levi factor of the stabilisor of x. Then the quiver $Q_{G/P}$ of G/P is isomorphic with the Hasse diagram $H_{G/P}$ of the Λ-module $\widehat{T_xX}/L_x$.*

If $G/P \subset V$, recall that $\widehat{T_xG/P} \subset V$ is the affine tangent space at x; it contains the line $L_x \subset V$ represented by $x \in \mathbb{P}V$, so that it makes sense to consider the quotient $\widehat{T_xG/P}/L_x$. We have stated this result without proof in [CMP 06, proposition 7]. In this article I need the explicit isomorphism, this is why I sketch the proof, leaving details to the reader.

Proof : It is known that to each vertex of the quiver one can associate a root of G. In fact, choose a reduced expression $w_{G/P} = s_{\beta_1} \dots s_{\beta_N}$ and set

$$\alpha_i = s_{\beta_N} \circ \dots \circ s_{\beta_{i+1}}(\beta_i).$$

Since two different reduced expressions for $w_{G/P}$ only differ by commutation relations, it is easy to check that the induced map from the set of vertices of the quiver to the set of roots is well-defined (it does not depend on the reduced expression). In the following, we consider that a reduced expression is chosen, thus identifying this set of vertices with $[1, N]$.

For example, if G/P is a smooth 8-dimensional quadric, then its quiver, and the corresponding roots, are given above (here we have shitfted the indices, denoting $(\epsilon_0, \dots, \epsilon_4)$ a basis of the weight lattice of $Spin_{10}$). Note that the highest weight of the corresponding $Spin_{10}$-representation is ϵ_0, and that we recover the Hasse diagram of $Spin_8$ by considering the weights $\epsilon_0 - \alpha_i$.

By [Pe 06, proposition 4.9], we may reduce the proof of our proposition to the particular case of a quadric of any dimension, as above, because if there is an arrow $i \to j$ in the quiver of G/P, then i and j belong to a subquiver of $Q_{G/P}$ isomorphic with the quiver of a quadric. It is also possible (and probably shorter) to check directly in each case that if ω denotes the highest weight of

$\Gamma(G/P, \mathcal{O}(1))$, then the set $\{\omega - \alpha_i : 1 \leq i \leq N\}$ is exactly the set of weights of the tangent space at the base point of G/P, and that the bijection $i \mapsto \omega - \alpha_i$ is an isomorphism of quivers $Q_{G/P} \to H_{G/P}$. $\square$

Given $[w] \in W/W_P$, we associate the Schubert subvariety $C_{[w]} \subset G/P$ which is the B-orbit closure of $[w] \in G/P$. Assuming that w is the minimal length representative of its class, we choose a reduced decomposition of w, and this defines a subquiver Q_w of the quiver $Q_{G/P}$ which is an order ideal (this means that if $i \to j$ is an arrow in $Q_{G/P}$ and $i \in Q_w$, then $j \in Q_w$: see [Pe 06, proposition 4.5]). We can also consider the subset H_w of $H_{G/P}$ which elements are the weights of $w^{-1}.T_{[w]}C_{[w]} \subset T_{[e]}G/P$. The following proposition will be useful to compute the dual variety of $C_{[w]}$, because it describes the tangent bundle of $C_{[w]}$:

Proposition 4.4. *Under the isomorphism $Q_{G/P} \simeq H_{G/P}$ of proposition 4.3, we have $Q_w = H_w$.*

Proof : Recall that ω denotes the highest weight of $\Gamma(G/P, \mathcal{O}(1))$. All the weights of $\widehat{T_x G/P}/L_x$ are of the form $\omega + \alpha$, where α are all the roots not in $\mathfrak{p} = Lie(P)$ (therefore α is a negative root). All the weights of $T_{[w]}C_{[w]}$ are of the form $w.\omega + \beta$, with β a positive root. Therefore, if $\omega + \alpha$ is a weight of $w^{-1}.T_{[w]}C_{[w]}$, $w.\alpha$ must be a positive root. So α must be a negative root sent by w to a positive root. Denote $l(w)$ the length of w; there are $l(w)$ such roots, namely $\{-\alpha_i : 1 \leq i \leq l(w)\}$. Since $l(w)$ is also the dimension of $C_{[w]}$, it follows that H_w is exactly the set of weights of the form $\omega - \alpha_i, 1 \leq i \leq l(w)$, so the proposition follows. $\square$

We now consider the case of E_6/P_3. Let $[w] \in W/W_3$; we want to define a quiver Q_{E_6/P_3} and a subquiver Q_w which pictures the tangent bundle of $C_{[w]}$. Since E_6/P_3 is not cominuscule, the quiver defined as in [Pe 06, definition 3.2] is not well-defined (it depends on a reduced expression of w_{E_6/P_3}), and as we have already seen, the cotangent bundle T^*E_6/P_3 is no longer irreducible, so its Hasse diagram is not suitable neither.

But our luck is that for $f \in T^*_\alpha E_6/P_3$, $q(f)$ only depends on $\pi(f) \in L_\alpha \otimes \wedge^2 Q^*_\alpha$; therefore, what we care for is not really the conormal bundle of $C_{[w]}$, but rather its projection to the bundle $L \otimes \wedge^2 Q^*$. This is why we consider the following :

Definition 4.1.

- *Let $[e] \in E_6/P_3$ be the base point, and let Λ denote a Levi factor of P_3.*
- *Let Q_{E_6/P_3} denote the Hasse diagram of the Λ-module $L^*_{[e]} \otimes \wedge^2 Q_{[e]} \subset T_{[e]}E_6/P_3$.*
- *For $[w] \in W/W_3$ with w the minimal length representative, let $Q_w \subset Q_{E_6/P_3}$ denote the set of weights of $w^{-1}.T_{[w]}C_{[w]} \cap (L_{[e]} \otimes \wedge^2 Q^*_{[e]})$.*

Proposition 4.5. *For $[w] \in W/W_3$, $Q_w \subset Q_{E_6/P_3}$ is an order ideal.*

Proof : For $a \in \{-2,-1,0,1,2\}$, let $\mathfrak{g}_k \subset \mathfrak{e}_6$ denote $\bigoplus_{\substack{\alpha = \sum_j k_j \alpha_j \\ k_3 = a}} \mathfrak{g}_\alpha$ (by this I mean that the Cartan subalgebra is included in $\mathfrak{g}_0$). We have $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $Lie(P_3) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The tangent space $T_{[e]}E_6/P_3$ decomposes as $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$; let $\mathcal{P}$ denote the weights of $\widehat{T_{[e]}E_6/P_3}/L_{[e]}$ which are of the form $\omega + \alpha$, with α a root of $\mathfrak{g}_{-1}$ ($\mathcal{P}$ is also the set of weights of $L_{[e]} \otimes \wedge^2 Q^*_{[e]}$). I claim that w induces an increasing bijection between $\mathcal{P}$ and its image. The proposition follows from this claim because $Q_{[w]}$ is the set of weights of $w^{-1}.T_{[w]}C_{[w]}$ which are in $\mathcal{P}$; arguing as in the proof of proposition 4.4, this is the set of roots of $\mathfrak{g}_{-1}$ which are mapped to a positive root by w, and this is obviously an order ideal since w is increasing.

To prove the claim, we note that $L_{[e]} \otimes \wedge^2 Q^*_{[e]}$ is a minuscule Λ-representation, since Λ contains $SL_2 \times SL_5$. Therefore W_P permutes transitively the roots in $\mathfrak{g}_{-1}$. Let α_0 be the highest root of $\mathfrak{g}_{-1}$, let α_1, α_2 be roots of $\mathfrak{g}_{-1}$ and assume $\alpha_1 \leq \alpha_2$. We can find $w_1, w_2 \in W_P$ such that $\alpha_i = w_i.\alpha_0$, assume moreover that w_1, w_2 are minimal such elements. Since $\alpha_1 \leq \alpha_2$, we have $w_1 \geq w_2$ for the Bruhat order, so that we may assume that a w_2 is a product of reflexions appearing in a reduced expression of w_1. Since w is a minimal length representative in W/W_3, the product $w.w_1$ is still a reduced expression, so $w.w_1 \geq w.w_2$, and so $w.\alpha_1 \leq w.\alpha_2$, as claimed. □

Remark : The same proof works for any G/P, as soon as $\mathfrak{g}_{-1}$ is a minuscule Λ-representation, with the notations of the proof.

4.3 Schubert varieties and dual varieties

The usual dual variety of a linear subspace is again a linear subspace. The goal of this section is to generalise this result for Schubert varieties.

Proposition 4.6. *Let $X \subset G/P$ be a suitable Schubert variety. Then $X^Q \subset G/Q$ is a Schubert variety.*

Proof : In fact, X^Q is a B-stable (proposition 1.3) irreducible closed subvariety of G/Q. □

Recall that B-stable Schubert varieties in G/P are parametrised by the quotient set W/W_P. For $[w] \in W/W_P$ (resp. $[x] \in W/W_Q$), we denote as in the previous subsection $C_{[w]} = \overline{B.[w]} \subset G/P$ the corresponding Schubert subvariety (resp. $D_{[x]} = \overline{B.[x]} \subset G/Q$). In the rest of this article, I give a description of the $[w]$'s such that the $C_{[w]}$ is suitable, and of the element in W/W_Q corresponding to the dual Schubert variety, in the fundamental cases. According to section 2, this is enough to describe all dual varieties of Schubert varieties. The strategy for this description is first to use a T-fixed point argument, to reduce the task to a purely combinatorial one. In the types A and D, I then give an explicit solution of this combinatorial problem. For the exceptional cases, my description of the dual Schubert varieties is not really explicit, but to compute them there is in principal only a finite number of computations to make.

So we fix the minimal G-representation V such that $G/P \subset \mathbb{P}V$. We denote $V = \oplus V_\lambda$ (resp. $V^* = \oplus V^*_\mu$) the weight decomposition of V (resp. V^*). Let

$C_{[w]} = \overline{B.[w]}$ be a Schubert variety. Recall that $\overline{N_{[w]}C_{[w]}}$ denotes the variety of y's in G/Q which are tangent to $C_{[w]}$ at $[w]$, see definition 3.6.

Lemma 4.1. *Let $[x] \in W/W_Q$ such that $C^Q_{[w]} = \overline{B.[x]}$. We have $[x] \in \overline{N_{[w]}C_{[w]}}$.*

Proof : First, notice that $C^Q_{[w]} = \overline{B.\overline{N_{[w]}C_{[w]}}}$. In fact, $B.\overline{N_{[w]}C_{[w]}}$ contains the set of y's in G/Q which are tangent at a point in $B.[w]$, therefore at a generic point of $C_{[w]}$.

Let μ_0 be the highest weight of V^* and denote $\mu = x.\mu_0$. Let $y \in V^*$ such that $[y] \in C^Q_{[w]} \subset \mathbb{P}V^*$. Use the weight decomposition of V^* to write $y = \sum_{\mu'} y_{\mu'}$. Since $[y] \in C^Q_{[w]}$ which is the closure of the B-orbit of the weight line of weight μ, $y_{\mu'} = 0$ if $\mu' \not\geq \mu$. Assume that $\forall [y] \in \overline{N_{[w]}C_{[w]}}, y_\mu = 0$. It would then follow from the first point that $\forall y \in C^Q_{[w]}, y_\mu = 0$, contradicting $[x] \in C^Q_{[w]}$.

Therefore, there exists $[y]$ in $\overline{N_{[w]}C_{[w]}}$ such that $y_\mu \neq 0$. Since $\overline{N_{[w]}C_{[w]}}$ is T-stable and V_μ is one-dimensional, we have $[x] \in \overline{N_{[w]}C_{[w]}}$. $\square$

I now explain how to compute the element $[x]$ of the previous lemma. We want to take into account the case of E_6/P_3, which cotangent bundle is not irreducible. Recall that in this case there is a natural bundle morphism $\pi : T^*E_6/P_3 \to L \otimes \wedge^2 Q^*$. To have uniform notations, in the other cases we denote $\pi : T^*G/P \to T^*G/P$ the identity and $\overline{q} = q$.

Decompose $\pi(T^*_{[e]}G/P)$ as a sum of weight spaces for the action of a Levi subgroup of P : $\pi(T^*_{[e]}G/P) = \oplus T^*_\tau$, and write similarly $\langle I_{[e]} \rangle = \oplus_\nu N_\nu$, where if $[w'] \in W/W_P$, $\langle I_{[w']} \rangle \subset V^*$ denotes the linear span of the Schubert variety $I_{[w']} \subset G/Q \subset \mathbb{P}V^*$ (see notation 1.9). It can be easily checked directly on the examples that all the weight spaces T^*_τ and N_ν have dimension 1. The rational map $\overline{q} : \pi(T^*_{[e]}G/P) \dashrightarrow G/Q \subset \mathbb{P}V^*$ is given by a list of polynomial functions $\pi(T^*_{[e]}G/P) \to N_\nu$ of the same degree d and with values in the complex line N_ν. The polarisations of these polynomials yield d-linear maps $T^*_{\tau_1} \times \cdots \times T^*_{\tau_d} \to N_\nu$, which will be denoted $P_{\tau_1,\ldots,\tau_d;\nu}$; remark that the space of such d-linear maps has dimension 1. Given $w \in W$, we denote $\mathcal{P}_w$ the set of weights ν such that there exist weights $\tau_1, \ldots, \tau_d$ of $\pi(w^{-1}.N^*_{[w]}C_{[w]}) \subset \pi(T^*_{[e]}G/P)$ such that $P_{\tau_1,\ldots,\tau_d;\nu}$ does not vanish.

Proposition 4.7. *Let $[w] \in W/W_P$, with w its minimal length representative. The variety $C_{[w]}$ is suitable if and only if $\mathcal{P}_w$ is not empty. In this case, if we denote $[x] \in W/W_Q$ such that $C^Q_{[w]} = C_{[x]}$, then $x.\mu_0$ equals $w.\mu_1$, where μ_1 is the lowest weight in $\mathcal{P}_w$.*

Proof : In fact, by lemma 4.1 and its proof, $x.\mu_0$ is the lowest weight μ, if any, such that the μ-component of the restriction of the rational map $\pi(T^*_{[w]}G/P) \dashrightarrow G/Q$ to $N^*C_{[w]}$ does not vanish identically.

The weights of $\langle I_{[w']} \rangle$ for $[w'] \in W/W_P$ are some weights of V^*, a set on which W acts; therefore it makes sense to talk of $w''.\mu'$, for $w'' \in W$ and μ' a weight of $\langle I_{[w']} \rangle$. I claim that w induces an increasing bijection between the weights of $\langle I_{[e]} \rangle$ and those of $\langle I_{[w]} \rangle$. The argument is similar to that of proposition 4.5. In fact, a weight of $\langle I_{[e]} \rangle$ can be written as $v.\mu_0$, with $v \in W_P$ and μ_0 the highest weight of V^*. Given two such weights $v_1.\mu_0 \geq v_2.\mu_0$, we can assume that v_2 is the minimal length representative of its class in $W_P/W_{P\cap Q}$. Thus v_1 can be written as a product of some reflections which occur in a reduced expression of v_2. Since $v_2 \in W_P$ and w is a minimal length representative, $l(wv_2) = l(w) + l(v_2)$. Therefore $wv_1 \leq wv_2$ for the Bruhat order, and thus $w.(v_1.\mu_0) \geq w.(v_2.\mu_0)$. This proves the claim.

Since the rational map $\overline{q} : \pi(T^*G/P) \dashrightarrow G/Q$ is G-equivariant, the weight μ of the proof of lemma 4.1 is also $w.\mu_1$, where μ_1 is the lowest weight such that the μ_1-component of the restriction of the rational map $T^*_{[e]}G/P \dashrightarrow G/Q$ to $w^{-1}.N^*_{[w]}C_{[w]}$ does not vanish identically. Obviously μ_1 is the lowest weight ν such that some $P_{\tau_1,\ldots,\tau_d;\nu}$ with $\tau_1,\ldots,\tau_d$ weights of $w^{-1}.N^*_{[w]}C_{[w]}$, does not vanish, so the proposition is proved. $\square$

We illustrate our method with the easy example $G/P = \mathbb{P}V$. Let $(e_1,\ldots,e_n)$ be a basis of V, let k be an integer and let $L_k = Vect(e_1,\ldots,e_k)$, $M_k = Vect(e_{k+1},\ldots,e_n)$. We consider the Schubert variety $X = \mathbb{P}L_k \subset \mathbb{P}V$ and compute its dual variety. The corresponding element of the Weyl group is the transposition $w = (1k)$. We have $T_{[w]}X \simeq Hom(e_k, L_k/e_k)$, so $w^{-1}.T_{[w]}X \simeq Hom(e_1, L_k/e_1)$ and so $w^{-1}.N^*_{[w]}X \simeq Hom(M_k, e_1)$. Since q is defined taking the kernel, the lowest weight in $q(w^{-1}.N^*_{[w]})$ is $\mu_1 = -\epsilon_{k+1}$. We have $w.\mu_1 = \mu_1$, so that the dual variety of X is the B-orbit closure of e^*_{k+1}, as expected. Note that in this example it would have been easier to compute directly the lowest weight in $q(N^*_{[w]}C_{[w]})$, instead of applying first w^{-1} and then w. In fact, this is what we will do to compute dual varieties of Schubert varieties in the cases $G/P = \mathbb{G}(r,V)$ and $G/P = \mathbb{G}^+_Q(2p+1,4p+2)$.

Recall from subsection 4.2 the definition of height of a vertex of the quiver $Q_{G/P} = H_{G/P}$. We denote h_0 the maximal h such that there exist $\tau_1,\ldots,\tau_d \in H_{G/P}$, ν a weight of $I_{[e]}$ such that $h(\tau_i) \geq h$ and $P_{\tau_1,\ldots,\tau_d;\nu} \neq 0$. We have the following values for h_0 (I have also indicated the height $h_{\max}$ of the heighest element of $Q_{G/P}$) :

G/P	$\mathbb{G}(r,n)$	$\mathbb{G}^+_Q(2p+1,4p+2)$	E_6/P_1	E_6/P_3
$h_{\max}$	$n-1$	$4p-1$	11	8
h_0	$\max(r,n-r)$	$2p+1$	8	5

Theorem 4.1. *The Schubert subvariety $C_{[w]}$ is suitable if and only if all the vertices of Q_w have height at most $h_0 - 1$.*

Proof : Unfortunately, I don't know how to prove in a uniform way this theorem. It will follow from propositions 4.8, 4.9, 4.10 and 4.11. The proof of these propositions also imply the above given values of h_0. $\square$

4.4 Case of Grassmannians

Recall that V is an n-dimensional vector space. We will parametrise Schubert varieties in $\mathbb{G}(r,V)$ by increasing lists of r integers, instead of partitions, because duality will appear easier to formulate in this way. The list (l_i) will correspond to the Schubert variety $C_l \subset \mathbb{G}(r,V)$ (resp. in $D_l \subset \mathbb{G}(r,V^*)$) which is the B-orbit closure of the linear space spanned by the l_i's T-eigenvectors in V (resp. in V^*). For $x \in \{1,\ldots,n\}$, we will write $x \in l$ to mean that there exists i such that $x = l_i$. The T-fixed points in V (resp. V^*) will be denoted e_i (resp. e^*_i). The T-fixed point whose B-orbit is dense in C_l (resp. D_l) will be denoted x_l

(resp. y_l). The Bruhat order on Schubert cells is given by $l \leq m$ if and only if $\forall i, l_i \leq m_i$. If $x_1, \ldots, x_r$ are distinct integers not necessarily increasing, we denote the list obtained reordering the x_i as $[x_1, \ldots, x_r]$.

Let T_l denote $Vect(e_i : i \in l)$ and let Q_l denote $Vect(e_i : i \notin l)$. The tangent space at x_l identifies with $Hom(Q_l, T_l)$. A weight in this space is given by a couple $(x, y) : x \in l, y \notin l$. Recall that the rational map

$$T^*_{x_l}\mathbb{G}(m,n) \simeq Hom(Q_l, T_l) \dashrightarrow \mathbb{G}(r, V^*)$$

is given by $\varphi \mapsto \ker \varphi$. Thus the degree of q is r and with the notations before proposition 4.7, we have :

Fact 4.2. *The multilinear form $P_{(x_1,y_1),\ldots,(x_r,y_r);l'}$ does not vanish if and only if the x_i's and the y_i's are all distinct, and l' is the set of the y_i's.*

Given a list l, we consider the list l^* defined inductively by

$$l^*_i = \min\{y : y > y_{i-1}, y > x_i, \forall j, y \neq x_j\}.$$

Lemma 4.3. *We have $\forall i, l^*_i \leq n$ if and only if $\forall i \in \{1, \ldots, r\}, l_i < n + 2i - 2r$.*

In terms of partitions, this means that the i-th part must be at least $r + 1 - i$.

Proof : Let i be an integer. The integers l_j for $j > i$ and l^*_j for $j \geq i$ are strictly greater than l_i and distinct, so the lemma follows. □

As the following proposition shows, $l \mapsto l^*$ is the combinatorial model for the duality of Schubert varieties in Grassmannians :

Proposition 4.8. *C_l is suitable if and only if $\forall i \in \{1, \ldots, r\}, l_i < n + 2i - 2r$. If C_l is suitable then $C^Q_l = D_{l^*}$.*

Proof : With the previous notations, the weights of the conormal space $N^*_{x_l}C_l$ are the couples (x, y) with $x \in l, y \notin l$, and $y > l_x$.

By proposition 4.7 and the comment after it, C_l is suitable if and only if there are lists $[y_1, \ldots, y_r]$ and $x_1, \ldots, x_r$ with (x_i, y_i) a weight of $N^*_{x_l}C_l$ and $P_{(x_1,y_1),\ldots,(x_r,y_r);l'} \neq 0$. In this case, if we denote l' the list such that $C^Q_l = D_{l'}$, then l' is the minimal possible such list. Moreover, in order that $P_{(x_1,y_1),\ldots,(x_r,y_r);l'} \neq 0$, all x_i must be distinct and we must have $\{x_i\} = l$, so we may assume by symmetry that $x_i = l_i$. It is easy to check that the set of such l' is not empty if and only if $\forall i \in \{1, \ldots, r\}, l_i < n + 2i - 2r$. In fact, if $l_i \geq n + 2i - 2r$, then the values y_j and x_j for $i \leq j \leq r$ must be distinct and between $n + 2i - 2r$ and n, a contradiction. Conversely, if $\forall i, l_i < n + 2i - 2r$, one may choose $y_i = l^*_i$.

We now show that l^* is indeed the minimal list. Let $[y_1, \ldots, y_r]$ be any list with $\forall i, y_i > l_i$ and $\forall i, j, y_i \neq l_j$. Let $(z_1, \ldots, z_r)$ be the corresponding ordered list (ie $\{y_1, \ldots, y_r\} = \{z_1, \ldots, z_r\}$ and $z_1 < z_2 < \cdots < z_r$). Then we have $z_1 > l_1$ and $\forall j, z_1 \neq l_j$, so $z_1 \geq l^*_1$. Say $z_i = y_{\sigma(i)}$. If $z_1 < l_2$ then $\sigma(1) = 1$. Thus in any case $z_2 > l_2$, so $z_2 \geq l^*_2$. By induction it follows that $\forall i, l^*_i \leq z_i$, so l^* is the minimal possible list, and proposition 4.7 finishes the proof. □

We illustrate this proposition with two examples. The array below computes two dual varieties in $\mathbb{G}(3, 8)$. It pictures the fact that for $l = (2, 4, 5)$ we have $l^* = \lambda = (3, 6, 7)$, and that for $m = (2, 4, 6)$ we have $m^* = \mu = (3, 5, 7)$:

	l	λ					
			l		λ		
				l		λ	

	m	μ					
			m	μ			
					m	μ	

Note that we have $C_l \subset C_m$ but we don't have $D_{l^*} \supset D_{m^*}$: contrary to the case $G/P = \mathbb{P}V$, duality of Schubert cells is no longer decreasing.

4.5 Case of spinor varieties

Schubert cells in $\mathbb{G}_Q^+(2p+1, 4p+2)$ (resp. $\mathbb{G}_Q^-(2p+1, 4p+2)$) are parametrised by lists of $+$ and $-$ signs, with an odd number of $+$ (resp. $-$) signs. The generic T-fixed point corresponding to the list (η_i) is the subspace generated by $e_i^{\eta_i}$, and will be denoted x_η. Schubert cells are also parametrised by strict partitions of size $2p$ (or subsets of $\{1, \ldots, 2p\}$), the correspondance being that we set $x \in \lambda$ $(1 \leq x \leq 2p)$ if $\eta_{2p+1-x} = -$.

Definition 4.2.

- *If $(\eta_i), 1 \leq i \leq 2p+1$, is a sequence of signs and j is an integer, we denote $\varphi(\eta, j)$ the sequence η' of signs such that $\eta'_i = \eta_i$ for exactly all i's but j.*
- *A sequence $(\eta_i), 1 \leq i \leq 2p+1$, of signs is admissible if*

$$\forall i \in \{1, \ldots, p\}, \#\{j : 1 \leq j \leq 2i, \eta_j = +\} \geq i.$$

Assume that η is admissible :

- *If there exists $i \leq p+1$ such that $\#\{j \;:\; 1 \leq j \leq 2i-1, \eta_j = +\} = i-1$, then let i_0 be the minimal such i, and set $\eta^* = \varphi(\eta, 2i_0 - 1)$.*
- *Otherwise there exists i such that*

$$\forall k \geq i, \;\#\{j : j \leq k, \eta_j = +\} > \#\{j : j \leq k, \eta_j = -\}.$$

Let i_0 be the minimal such i and set $\eta^* = \varphi(\eta, i_0)$.

If there does not exist $i \leq p+1$ such that $\#\{j \;:\; 1 \leq j \leq 2i-1, \eta_j = +\} = i-1$, then $\#\{j \;:\; 1 \leq j \leq 2p+1\} \geq p+1$, so $i = 2p+1$ satisfies the condition of the last point of this definition.

Let η be fixed. Since the positive roots are $\epsilon_i \pm \epsilon_j$ with $i < j$, the restriction of the Bruhat order on the set of $\varphi(\eta, j)$ is given by :

Fact 4.4. *We have $\varphi(\eta, i) \leq \varphi(\eta, j)$ for the Bruhat order if and only if*

$$\text{or} \left\{ \begin{array}{l} \eta_i = \eta_j = + \text{ and } i \leq j \\ \eta_i = + \text{ and } \eta_j = - \\ \eta_i = \eta_j = - \text{ and } i \geq j. \end{array} \right.$$

Proposition 4.9. *C_η is suitable if and only if η is admissible, and in this case $C_\eta^Q = D_{\eta^*}$.*

Proof : Recall that $x_\eta \in G/P$ denotes the linear space spanned by $e_i^{\eta_i}$. It is well-known that $T_{x_\eta} G/P$ identifies with $\wedge^2 Vect(e_i^{\epsilon_i})^*$. Moreover, $N^*_{x_\eta} C_\eta \subset \wedge^2 Vect(e_i^{\epsilon_i})$ is generated by $e_i^{\eta_i} \wedge e_j^{\eta_j}$ for (i, j) such that $\eta_i = +$ and $i < j$. In fact, with the notations of [Bou 68, PLANCHE IV], the weight of x_η is $\rho_\eta = \frac{1}{2} \sum \eta_i \epsilon_i$, and the weights of $T_{x_\eta} C_\eta$ are the weights of the form $\frac{1}{2} \sum \eta'_i \epsilon_i$ which can be expressed as $\rho_\eta + \alpha$, where α is a positive root. Therefore the claim follows from the fact that the positive roots are $\epsilon_i \pm \epsilon_j$ with $i < j$.

The weights of $T^*_{x_\eta}G/P \simeq \wedge^2 Vect(e_i^{\epsilon_i})$ are parametrised by couples (x, y) of integers, with $x < y$. Now let $x_k, y_k, 1 \leq k \leq p$, be integers with $x_k < y_k$. With the notations of subsection 4.3, μ is given by a set of polynomials of degree p and the p-multilinear map $P_{(x_k,y_k);\eta'}$ does not vanish if and only if the x_k's and the y_k's are all distinct and $\eta'_i = \eta_i$ for exactly all i's which belong to the set $U := \{x_k\} \cup \{y_k\}$.

Given the previous description of $N^*_{x_\eta}C_\eta$, the Schubert variety C_η will be suitable if and only if we can find (x_k, y_k) such that

$$x_k < y_k, \text{ the } x_k, y_k \text{ are all distinct, and } \eta_{x_k} = +. \tag{2}$$

Therefore, for all i's with $1 \leq i \leq p$, we have the inequality

$$2i - 1 \leq \#(U \cap [1, 2i]) \leq 2\#\{j \ : \ 1 \leq j \leq 2i, \eta_j = +\}.$$

This implies that η should be admissible.

Conversely, assuming that η is admissible, let us consider the following algorithm which produces a list of distinct elements (x_k, y_k) with $\eta_{x_k} = +$ and $x_k < y_k$. If $\forall i > 1, \eta_i = +$, set $x_k = 2p$ and $y_k = 2p+1$. Otherwise, let i_0 be the minimal $i > 1$ such that $\eta_i = -$; set $x_1 = i_0 - 1$ (the fact that η is admissible garanties that even in the case $i_0 = 2$, we have $\eta_{i_0-1} = +$) and $y_1 = i_0$. Remove η_{x_1} and η_{y_1} from the list η : this new list is again admissible, as one checks readily. Therefore, it is possible to define (x_k, y_k) for $k \geq 2$ inductively.

We therefore have proved that C_η is suitable if and only if η is admissible. Let us now compute the dual variety. Assume first that there exists $i \in \{1, p+1\}$ such that

$$\#\{j : 1 \leq j \leq 2i - 1, \eta_j = +\} = i - 1. \tag{3}$$

Let i_0 be the minimal such i. Admissibility of η implies that $\#\{j \ : \ 1 \leq j \leq 2i_0 - 2, \eta_j = +\} = i_0 - 1$ and so $\eta_{2i_0-1} = -$. Therefore, if (x_k, y_k) is any sequence satisfying (2) and $U = \{x_k\} \cup \{y_k\}$, there exists $j \leq 2i_0 - 1$ such that $\eta_j = -$ and $j \notin U$. Thus if there exists (x_k, y_k) such that $P_{(x_k,y_k);\eta'} \neq 0$ for some η', this implies $\eta' \geq \varphi(\eta, 2i_0 - 1)$ (recall fact 4.4). Conversely, the previous algorithm produces a sequence (x_k, y_k) for which it is easy to see that $P_{(x_k,y_k);\varphi(\eta,2i_0-1)} \neq 0$. Thus $\eta^* = \varphi(\eta, 2i_0 - 1)$ is the lowest list one can obtain in this way, so that $C^Q_\eta = D_{\eta^*}$ as claimed in this case.

Assume finaly that (3) holds for no $i \in \{1, p+1\}$. Therefore, as we have seen, there exists i (for example $i = 2p+1$) such that

$$\forall k \geq i, \#\{j \ : \ j \leq k, \eta_j = +\} > \#\{j \ : \ j \leq k, \eta_j = -\}.$$

Let i_0 be the minimal such i. Obviously, if i is any integer, and (x_k, y_k) satisfies (2) and $x_k, y_k \neq i$, then

$$\forall k \geq i, \#\{j \ : \ j \leq k, \eta_j = +, j \neq i\} \geq \#\{j \ : \ j \leq k, \eta_j = -, j \neq i\},$$

so that $i \geq i_0$. So $P_{(x_k,y_k);\eta'} \neq 0$ implies $\eta' \geq \varphi(\eta, i_0)$. Again, the explicit algorithm provides a sequence (x_y, y_k) such that $P_{(x_k,y_k);\varphi(\eta,i_0)} \neq 0$, and therefore $D_{\varphi(\eta,i_0)}$ is the dual variety of C_η. $\square$

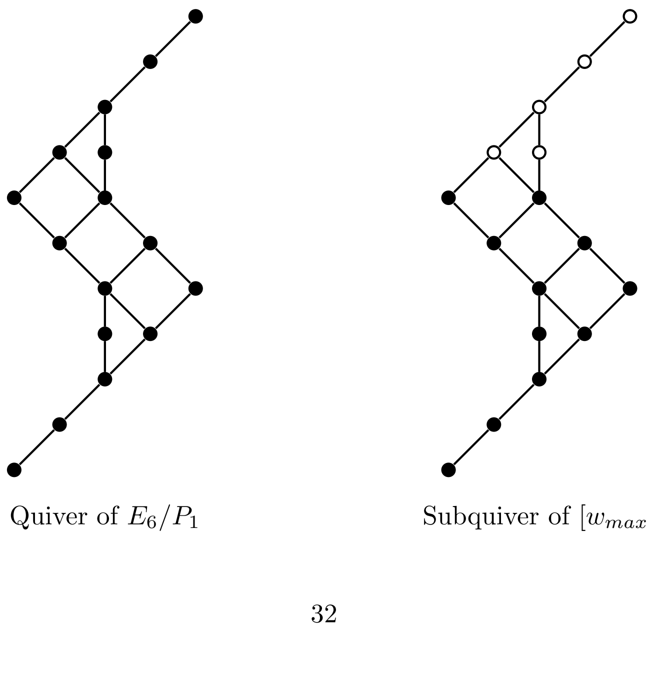
4.6 Case of $E_{6,I}$

We now consider the exceptional cases when G is of type E_6. Recall that there are two possibilities for (P,Q) : either they correspond to the roots (α_1,α_6) or (α_3,α_5). In each case I explain in which case $P_{\tau_1,\ldots,\tau_d;\nu}$ does not vanish. Using subsection 4.3, this describes in principle all dual varieties to Schubert varieties, although I will not give a simple combinatorial recipy for this correspondance (note however that to give such a description there is "only" a finite number of computations to do). My description of which $P_{\tau_1,\ldots,\tau_d;\nu}$ don't vanish will however yield a simple caracterisation of the suitable Schubert varieties.

As we have seen in subsection 1.2, a Levi factor L of P_1 is isomorphic with $\mathbb{C}^* \times Spin_{10}$, and $T_{[e]}E_6/P_1$ identifies with a 16-dimensional spinor representation of L. Moreover, the closed L-orbit in $\mathbb{P}T^*_{[e]}G/P$ identifies with a L-homogeneous spinor variety : it is a connected component of the variety parametrising isotropic linear spaces of dimension 5 in a certain quadratic vector space of dimension 10 that we will denote M. It is proved in [Ch 06, corollary 3.2] that $I_{[e]} \subset \mathbb{P}M$ is the corresponding 8-dimensional quadric acted upon by L and that the rational map $T^*_{[e]}E_6/P_1 \dashrightarrow I_{[e]}$ is induced by the unique L-equivariant quadratic map $T^*_{[e]}G/P \to M$. The polarisation $\mathcal{P}$: $T^*_{[e]}E_6/P_1 \times T^*_{[e]}E_6/P_1 \to M$ of this equivariant map has the following geometric interpretation : for $x,y \in T^*_{[e]}E_6/P_1$ representing points of the spinor variety corresponding to the isotropic linear spaces L_x, L_y, the class of $\mathcal{P}(x,y)$ in $\mathbb{P}M$ is the intersection of L_x and L_y if this intersection has dimension 1, and $\mathcal{P}(x,y)=0$ otherwise.

Denote as in subsection 4.5 $(e_1^+,\ldots,e_5^+,e_1^-,\ldots,e_5^-)$ a base of M such that the quadratic form Q satisfies $Q(\sum x_i^+e_i^+ + \sum x_i^-e_i^-) = \sum x_i^+x_i^-$. An L-weight of M can therefore be denoted $\nu \in \{1^+,\ldots,5^+,1^-,\ldots,5^-\}$, and a weight $\eta = (\eta_1,\ldots,\eta_5)$ in $T^*_{[e]}G/P$ corresponds to a list of plus or minus signs, with an odd number of plus signs. The condition for $P_{\eta,\eta';\nu}$ not to vanish is thus that η and η' have exactly one sign in common which is ν.

From this description we can describe the suitable Schubert varieties. In the array below, I recall the quiver of E_6/P_1 and I define an element $[w_{max}] \in W/W_P$ by its subquiver :

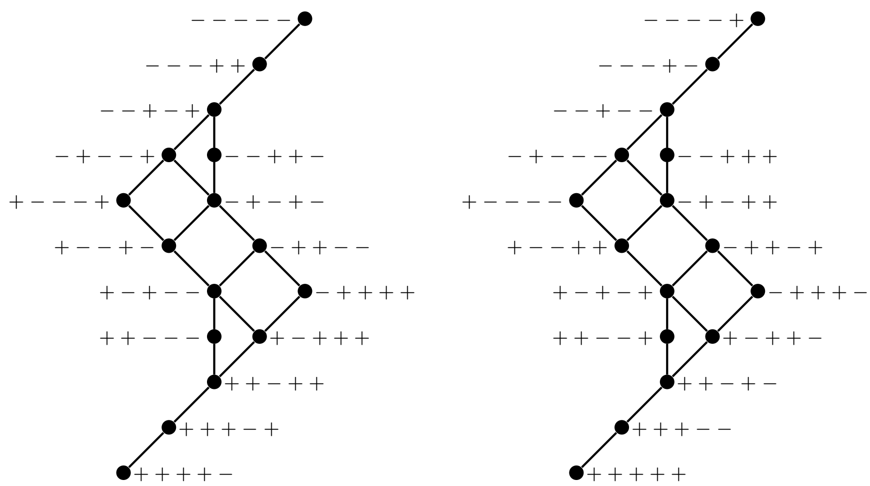


Quiver of E_6/P_1 Subquiver of $[w_{max}]$

We have the following proposition :

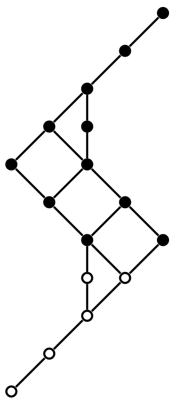
Proposition 4.10. *Let $[w] \in W/W_P$. Then the Schubert variety $C_{[w]}$ is suitable if and only if $[w] \leq [w_{max}]$.*

Proof : Below we give the Hasse diagram H (resp. H^*) of the L-module $T_{[e]}E_6/P_1$ (resp. $T^*_{[e]}E_6/P_1$), which, by proposition 4.3, is isomorphic with the quiver of E_6/P_1 :

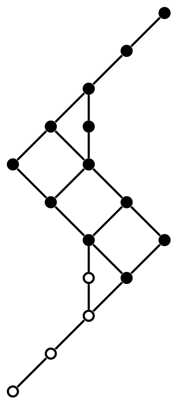


Let $\iota : H \to H^*$ be induced by the map $\eta \mapsto -\eta$ (in terms of quivers, this corresponds to the obvious symmetry). Let $[w] \in G/P$ and $Q_{[w]} \subset H$ the corresponding subquiver, marking the weights of $w^{-1}.T_{[w]}C_{[w]}$. Thanks to proposition 4.7, the proposition amounts to the fact that we can find two weights $\eta, \eta' \in H^* - \iota(Q_{[w]})$ and which have only one sign in common if and only if $Q_{[w]} \subset Q_{[w_{max}]}$. This may be seen as follows :

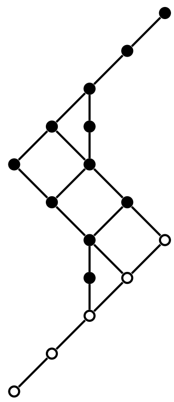
- If $Q_{[w]} \subset Q_{[w_{max}]}$, we can set $\eta = (+ + - - +)$ and $\eta' = (+ - + + -)$ to check that the corresponding Schubert variety is suitable (below the subset $\iota(Q_{[w_{max}]})$ is drawn).
- If $Q_{[w]}$ contains the vertex corresponding to the weight $(-+--+)$, $\iota(Q_{[w]})$ contains the subset drawn below. Thus η and η' are weights which begin with $++$, so they have two common signs. The corresponding Schubert variety is not suitable.
- The last case is that the subquiver contains the vertex corresponding to the weight $(- - + + -)$. Thus η and η' have at least 3 plus signs among the 4 first signs, and therefore have at least 2 common signs.



$\iota(Q_{[w_{max}]})$



$(-+--+) \in Q_{[w]}$



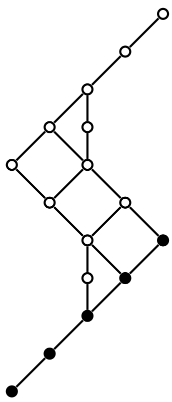
$(--++-) \in Q_{[w]}$

□

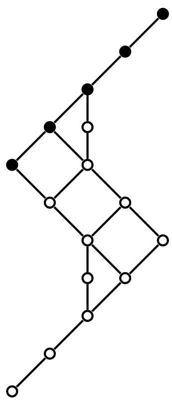
Example 4.3. *Let $X \subset E_6/P_1$ be a linear subspace of maximal dimension 5. Then $X^Q \subset E_6/P_6$ is also a linear subspace of dimension 5.*

Therefore, this provides, in our setting, a new example of a variety which is isomorphic to its dual. Similar examples in the usual setting $X \subset \mathbb{P}^n$ are projective subspaces, quadrics, $\mathbb{G}(2, 2p+1)$, and the spinor variety $\mathbb{G}_Q^+(5, 10)$.

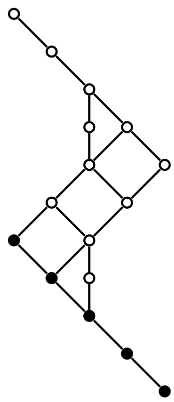
Proof : Let $X \subset E_6/P_1$ be a linear subspace of dimension 5. The variety parametrising linear subspaces of maximal dimension 5 is given by Tits shadows, according to [LM 03, theorem 4.3]. In particular, it is a homogeneous variety, and so we can assume that X is the Schubert variety corresponding to the Weyl group element $w = s_6 s_5 s_4 s_3 s_1$. The corresponding quiver Q_w and $\iota(Q_w)$ follow; we have also drawn the quiver Q_{w^*} of the dual variety.



Q_w



$\iota(Q_w)$



Q_{w^*}

According to proposition 4.7, we must look for two weights of H^* not in $\iota(Q_w)$ which have only one common sign. If this common sign is a minus sign, then among these two weights there are 6 minus signs. Therefore one of the weight has 4 minus signs which is impossible given $\iota(Q_w)$ and the Hasse diagram

H^*. Since $(--+++)$ and $(+++--+)$ are weights not in $\iota(Q_w)$, the lowest weight is 5^+. Note that this weight is obtained applying $w_6 = s_3s_4s_5s_6$ to the highest weight. Therefore, proposition 4.7 shows that the dual variety to X corresponds to the class of $w.w_6 = s_6s_5s_4s_3s_1s_3s_4s_5s_6$ modulo W_6. Modulo W_6, we have

$$\begin{array}{cccccc} & s_6s_5s_4s_3s_1s_3s_4s_5s_6 & = & s_6s_5s_4s_1s_3s_1s_4s_5s_6 & = & s_6s_5s_4s_1s_3s_4s_5s_6s_1 \\ \equiv & s_6s_5s_1s_4s_3s_4s_5s_6 & = & s_6s_5s_1s_3s_4s_3s_5s_6 & \equiv & s_6s_5s_1s_3s_4s_5s_6 \\ = & s_6s_1s_3s_5s_4s_5s_6 & \equiv & s_6s_1s_3s_4s_5s_6 & \equiv & s_1s_3s_4s_5s_6. \end{array}$$

This proves the claim. □

4.7 Case of $E_{6,II}$

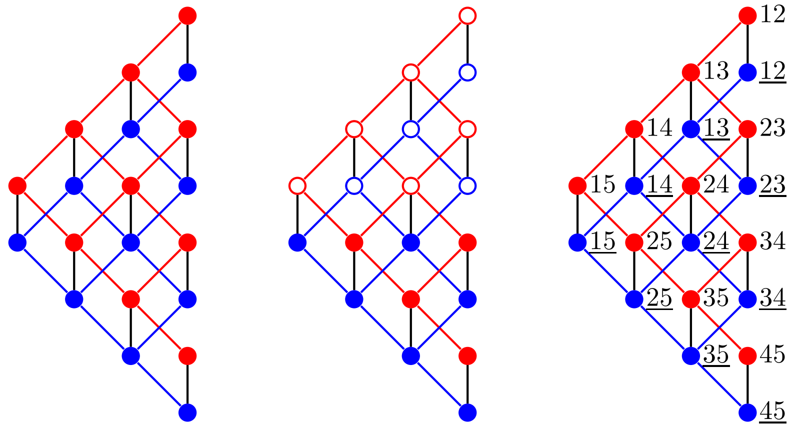
Let $\alpha \in E_6/P_3$ be the base point. Recall that there is a surjection $\pi : T^*_\alpha E_6/P_3 \to L_\alpha \otimes \wedge^2 Q^*_\alpha$ and that the rational map $q : T^*_\alpha E_6/P_3 \dashrightarrow I_\alpha$ is induced by a rational map $\overline{q} : L_\alpha \otimes \wedge^2 Q^*_\alpha \dashrightarrow I_\alpha = \mathbb{G}(2, Q^*_\alpha)$.

The description of $\overline{q}$ given in subsection 1.2 implies that $\overline{q}$ has degree 6. Consider as in subsection 4.3 its polarisation, with coordinates denoted P. In order to give a non-vanishing criterium for P, let us introduce some notation. Let e_1, e_2 be a basis of L_α and $f_1^*, \ldots, f_5^*$ a basis of Q^*_α. The weight of the vector $e_1 \otimes (f_i^* \wedge f_j^*)$ with $i < j$ will be denoted ij, and the weight of the vector $e_2 \otimes (f_i^* \wedge f_j^*)$ will be denoted $\underline{ij}$. Finally, the weight of $f_i^* \wedge f_j^*$ will be denoted ij^*.

Let $\tau_1, \ldots, \tau_6$ be weights of $L_\alpha \otimes \wedge^2 Q^*_\alpha$ and ν a weight of $\wedge^2 Q^*_\alpha$, if $P_{\tau_1,\ldots,\tau_6,\nu} \neq 0$, then $\#\{k : \tau_k \in \{ij\}\} = \#\{k : \tau_k \in \{\underline{ij}\}\} = 3$. So we assume that this is the case and that τ_1, τ_2, τ_3 (resp. τ_4, τ_5, τ_6) are of the form ij (resp. $\underline{ij}$).

With this setting, $P_{i_1j_1,i_2j_2,i_3j_3,\underline{k_1l_1},\underline{k_2l_2},\underline{k_3l_3};mn^*}$ will not vanish if and only if, up to permuting the three first weights and the three last, we have that i_1, j_1, i_2, j_2 (resp. k_1, l_1, k_2, l_2) are all dinstinct; say they take all values in $\{1, \ldots, 5\}$ except u (resp. v). Moreover we must have $u \in \{k_3, l_3\}$ (resp. $v \in \{i_3, j_3\}$), say $\{k_3, l_3\} = \{u, u'\}$ (resp. $\{i_3, j_3\} = \{v, v'\}$). Finally, we must have $u' \neq v'$ and $\{u', v'\} = \{m, n\}$.

In principle, this combinatorial rule describes dual varieties of Schubert cells in this case. However, as in the case of $E_{6,I}$, we can be more precise as far as suitability is concerned. The Hasse diagram H of $L^*_\alpha \otimes \wedge^2 Q_\alpha$ is given below, as well as a subquiver denoted Q_{max}. I have also indicated the Hasse diagram H^* of $L_\alpha \otimes \wedge^2 Q^*_\alpha$:



H $\qquad\qquad$ Q_{max} $\qquad\qquad$ H^*

In these pictures and the following, the weights ij are drawn in red and the weights $\underline{ij}$ are drawn in blue. Given a Schubert cell $C_{[w]}$, with w the minimal length representative of $[w] \in W/W_P$, recall that we associated (not injectively) to this Schubert cell a subquiver $Q_{[w]}$ of H in subsection 4.2.

Proposition 4.11. *The Schubert cell $C_{[w]}$ is suitable if and only if we have $Q_{[w]} \subset Q_{max}$.*

Proof : As in the preceeding subsection, we define $\iota : H \to H^*$ given by $\eta \mapsto -\eta$. The weights of H^* have been given above.

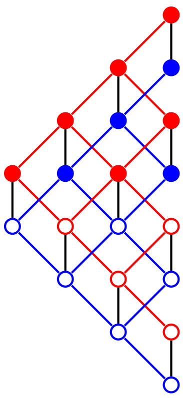
Since q is induced by $\overline{q}$, a Schubert variety $C_{[w]}$ will be suitable if and only if the rational map $\overline{q} : L_\alpha \otimes \wedge^2 Q_\alpha^* \dashrightarrow I_\alpha$ is defined generically on $\pi(w^{-1}.N^*_{[w]}C_{[w]})$; equivalently, $\overline{q}$ should be defined on the orthogonal of $w^{-1}.T_{[w]}C_{[w]} \cap L_\alpha^* \otimes \wedge^2 Q_\alpha$ in $L_\alpha \otimes \wedge^2 Q_\alpha^*$. Equivalently again, we should be able to find 6 weights τ_k not in $\iota(Q_{[w]})$ and some integers i, j such that $P_{\tau_k;ij^*}$ does not vanish.

In case $Q_{[w]}$ is included in Q_{max}, we can consider the weights $34, 25, 34, \underline{15}$, $\underline{24}$, $\underline{15}$ (the corresponding subset $\iota(Q_{max})$ is drawn below), which satisfy the relation $P_{34,25,34,\underline{15},\underline{24},\underline{15};45^*} \neq 0$ and do not belong to $\iota(Q)$. Otherwise, there are four cases :

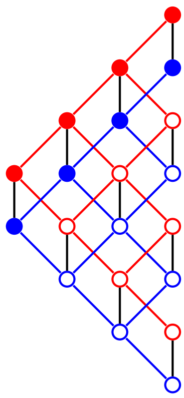
- If $Q_{[w]}$ contains the weight 15, by proposition 4.5, $\iota(Q_{[w]})$ contains the corresponding subset in the array below. The remaining weights are of the form ij or $\underline{ij}$ with $1 < i < j$, so the Schubert variety cannot be suitable.
- If $Q_{[w]}$ contains the weight $\underline{14}$, the remaining weights are of the form ij or $\underline{kl}$ with $2 < i < j$ (see the array below), so again the Schubert variety cannot be suitable.
- If $Q_{[w]}$ contains the weight 24, let $i_1j_1, i_2j_2, i_3j_3, \underline{i_4j_4}, \underline{i_5j_5}, \underline{i_6j_6}$ be a list of weights not in $\iota(Q_{[w]})$. Note that we have $i_kj_k \in \{12, 13, 14, 15, 34\}$ for all k. Assume that there exists kl^* such that $P_{i_1j_1,i_2j_2,i_3j_3,\underline{i_4j_4},\underline{i_5j_5},\underline{i_6j_6};kl^*} \neq 0$. This implies that the only integer x (resp. y) which does not belong to the set $\{i_1, j_1, i_2, j_2\}$ (resp. $\{i_4, j_4, i_5, j_5\}$) is either 2 or 5. This integer

must therefore belong to the set $\{i_3, j_3\}$ (resp. $\{i_3, j_3\}$), so we must have $\{i_3, j_3\} = \{1, x\}$ (resp. $\{i_6, j_6\} = \{1, y\}$). This implies that $k = l = 1$, a contradiction.

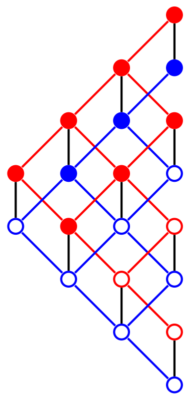
- Assume finaly that $Q_{[w]}$ contains the weight $\underline{23}$. In this case all the weights which are not in $\iota(Q_{[w]})$ and of the form ij satisfy $j = 5$. Again, the Schubert variety is not suitable.



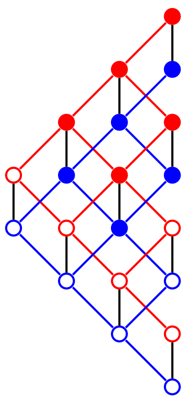
Case $\iota(Q_{max})$



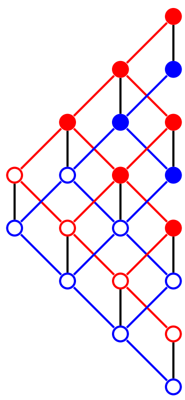
Case $15 \in Q_{[w]}$



Case $\underline{14} \in Q_{[w]}$



Case $24 \in Q_{[w]}$



Case $\underline{23} \in Q_{[w]}$

$\square$

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